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Ph.D. Thesis

Geometric Lang-Vojta Conjecture in $\mathbb{P}^2$

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Knowing,
my most esteemed friend Dionysius,
that you are anxious to learn
how to investigate problems in numbers,
I have tried [. . .] to set forth to you
the nature and power
subsisting in numbers.

Diophantus - *Arithmetica*

Abstract

Lang-Vojta conjecture is one of the most celebrated conjectures in Diophantine Geometry. Stated independently by Paul Vojta in [Voj1] and Serge Lang (see [Lan3]), the conjecture predicts degeneracy of S-integral points in algebraic varieties of log-general type for a finite set of places S of a number field κ containing the infinite ones, provided that the divisor "at infinity" is a normal crossing divisor. This deep conjecture and his analogous formulations are among the main open problems in Number Theory, Complex Analysis and Arithmetic Algebraic Geometry.

This thesis contains the work of the author during his Ph.D. studies at the University of Udine under the supervision of Prof. Pietro Corvaja (and, partially, during his visit to Brown University under the supervision of Prof. Dan Abramovich), and it is centered around the function field version of Lang-Vojta conjecture for complements of curves in $\mathbb{P}^2$, with at most normal crossing singularities. The main part contains the proof of two cases of this conjecture, namely the non-split case for complements of degree four and three components divisors and the split case for very generic divisors of degree four with simple normal crossing.

After a detailed review of the main conjecture, to which is devoted the first part, in the second part we deal with an extension of previous results of Corvaja and Zannier [CZ5] for complements of a conic and two lines to the so-called non-

split case. We focus on the situation of an affine threefold fibered over a curve and we study sections of fibrations in which every fiber is isomorphic to the complement of a conic and two lines in $\mathbb{P}^2$. We describe completely the moduli of degree four and three components divisors in the projective plane; this permits us to link the problem to the study of solutions to a certain equation with non constant coefficients in the function field of the base curve. Then we implement methods relying on S-unit gcd for function fields and height considerations already available for the split case and we adapt them to our case obtaining algebraic hyperbolicity for sections of these fibrations.

The third part is dedicated to the case in which the divisor at infinity has less than three irreducible components. We adopt a new strategy to tackle this problem based on ideas coming from logarithmic geometry. We begin extending previous result for the three component case in the language of logarithmic stable maps. Using this extension we reformulate the problem as vanishing of certain moduli space of minimal stable maps $\mathcal{K}_\Gamma(\mathbb{P}^2, D)$ in the sense of [AC]. Finally we use the properness of the stack $\mathcal{K}_\Gamma(\mathbb{P}^2, D)$ of minimal logarithmic stable maps to the Deligne-Faltings pair $(\mathbb{P}^2, D)$ in order to obtain algebraic hyperbolicity for complements of very generic quartics D deforming to a conic and two lines in $\mathbb{P}^2$. This provides first examples of Lang-Vojta conjecture for complement of both irreducible quartics and two conics, although a very generic hypothesis is needed.

Acknowledgments

Traveler, the road is only
your footprint, and no more;
traveler, there's no road,
the road is your traveling.
Going becomes the road
and if you look back
you will see a path
none can tread again.

Antonio Machado - Proverbios y cantares (XXIX)

The journey to the end of this Thesis has been long and challenging but I had the luck of sharing it with so many supporting and inspiring people. It is a pleasure to thank (some of) them who helped me, knowingly and unknowingly, in the successful completion of this project.

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I had the fortune to share my office with friends more than colleagues: to Daniele, Edda, Emanuele and Luca go my gratitude for laughs, shared (and divided) food, happy moments, coffees and the (small amount of) math we made together in these years. At the same time my university life has been enriched with the presence of many good friends: Sara has been and still is the best friend I have ever met and Tommaso has always brought that lighthearted joy to my work days (although he tried to stop my Thesis with Minecraft©!).

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Contents

Introduction

Diophantine Geometry can be described as the study of integral or rational solutions to polynomial equations with integral or rational coefficients, the so-called *Diophantine Equations*, using geometric tools. In its simpler instance it asks for a "geometric" solution to the following

Problem (see [Mor2]) Find reasonably simple necessary conditions for the solvability of

$$f(x_1, \ldots, x_n) = 0$$

for a polynomial f with coefficients in $\mathbb{Q}$, with $(x_1, \ldots, x_n)$ in either $\mathbb{Z}^n$ or $\mathbb{Q}^n$.

The term "geometric" stands for depending on the *geometric* properties of the complex algebraic variety defined by f, i.e. the set of *complex* solutions to the previous equation. More precisely and more generally, fixed a field κ (which usually is either a number field, or a function field of an algebraic variety), we can describe Diophantine Geometry as the study of the set $X(\kappa)$ for an algebraic variety X over κ. Rephrasing Serge Lang, who coined the term for this discipline, one can say that the main goal of Diophantine Geometry is to determine geometric properties of the quasi-projective variety X that characterize the set of points $X(\kappa)$, e.g. that imply that $X(\kappa)$ is either non-empty or finite or dense with respect to the Zariski topology. Here *geometric* means properties that can be checked on the algebraic closure of κ.

While the study of Diophantine Equations is one of the oldest branches of mathematics which can be traced back to Babylonians and Egyptians, Diophantine Geometry is a fairly recent subject whose name comes from the seminal first edition of Lang's book [Lan3]. From its birth numerous results on the behavior of sets $X(\kappa)$ have been achieved, but at the same time, it seems that a large amount of open problems and conjectures are still out of reach.

One of the most intriguing and influential open problems in this field is to determine whether there exist infinitely many integral points on algebraic varieties defined over $\mathbb{Q}$, or equivalently infinitely many integral solutions to the system of equations defining the variety. Since the Siegel Theorem, and later Falting's proof of Mordell Conjecture, this task has been completed for the case of algebraic curves. In particular, it has been shown that whenever an affine curve has genus greater than zero it contains only finitely many integral points. For algebraic surfaces the problem turns out to be much more subtle and challenging and, although deep results have recently been obtained, a complete solution seems to be, at present, beyond hope. Nevertheless, a number of conjectures have been stated and serve as both focal points as well as the direction towards which Diophantine Geometry research is moving. Among these conjectures, one of the most important, and the one to which this Thesis is devoted, is the conjecture due to Paul Vojta [Voj1] and Serge Lang [Lan3] which for number fields (in the surface case) reads as follows:

Conjecture 1 (Lang-Vojta for number fields). *Let X be a smooth affine surface defined over a number field k. Let $\tilde{X}$ be a smooth projective variety containing X as an open subset. Let $D = \tilde{X} \setminus X$ be the divisor at infinity and $K = K_{\tilde{X}}$ be a canonical divisor of $\tilde{X}$. Suppose that D is a normal crossing divisor. Then if $D + K$ is big, for every ring of S-integers $\mathcal{O}_S \subset k$, the set of S-integral points $X(\mathcal{O}_S)$ is not Zariski-dense.*

The history of this Conjecture can be traced back to the work of Vojta in [Voj1], where he formulated several conjectures on distribution of integral, S-integral, and algebraic points on projective varieties which «seem to contain virtually all Diophantine statements that are currently proven or conjectured» [HS]. Independently Lang proposed a series of conjectures on the same themes conjecturing a unified description of the so-called *exceptional set* on log-general varieties over number and function fields and its analogous in Complex Analysis (see [Lan1], [Lan2], [Lan3] and [Lan4]). These two profound visions of Number Theory (and Complex Analysis) meet together in the aforementioned Conjecture which we called, as customary, with both names.

Conjecture 1 extends Bombieri-Lang Conjecture on rational points on general type varieties and can be regarded as one of the biggest open problems in Arithmetic Geometry. Unfortunately only some special cases of the Conjecture are known, and a complete solution at the present seems to be out of reach. In particular, almost all the results follows from Falting's Big Theorem, which proves a famous conjecture by Lang on the non-density of rational points for irreducible subvarieties of abelian varieties which are not a translate of an abelian subvariety.

A slightly easier problem to tackle is the one obtained by the previous Conjecture by replacing the number field k with a function field in one variable $\kappa(\mathcal{C})$, i.e. the function field of an algebraic curve $\mathcal{C}$, and the set S by a finite set of points of $\mathcal{C}$. In this setting, one can try to use geometric tools not available in the number field case, like the existence of derivatives and differential forms canonically defined on the curve $\mathcal{C}$, to obtain stronger and broader results. Moreover, the similarity of this approach with problems related to the distribution of entire curves in complex analytic manifolds allows one to borrow ideas and techniques from this related field.

In the last ten years, following a new proof of Siegel's Theorem using Schmidt's Subspace Theorem, Corvaja and Zannier were able to prove several special cases of Lang-Vojta's Conjecture in the case in which the divisor at infinity has sufficiently many irreducible components. Among these results, when the compactification of the affine surface X is the projective plane $\mathbb{P}^2$, they solve the Conjecture (the so-called *split case*) provided that the divisor D has at least three components. This seems to be the most general situation in which their method can be applied for complements of normal crossing divisors in $\mathbb{P}^2$. In particular this leaves open the case of log-irregularity strictly lesser than two. Using a different strategy, Xi Chen proved that the complement of a very generic plane curve, with at most normal crossing singularities, of degree at least 5, verifies Conjecture 1. His result, moving from different motivations, related to the famous Kobayashi Conjecture, is based on a very refined deformation argument which reduces the general situation to the known case for complements of hyperplanes in general position. Similarly as before, it seems that his techniques cannot be brought further to the complete proof of the Conjecture in the case $\tilde{X} = \mathbb{P}^2$.

The present Thesis focuses on two open problems related to these results. The first one is a generalization of Corvaja and Zannier's result for complements of a conic and two lines in general position in $\mathbb{P}^2$. In particular we considered Conjecture 1 in the non-split case, i.e. when the conic and the lines are not defined on the base field. This led to consider fibered threefolds in which every fiber is isomorphic to the complement of a degree four divisor with three irreducible components. In this situation we were able to prove Lang-Vojta Conjecture, and explicitly determine a bound for the degree of affine curves in terms of their Euler Characteristic. Moreover we calculate the dependency of this bound on the structure of the threefold, based on a description of the moduli space of the conic and two lines in $\mathbb{P}^2$.

The second problem we deal with in this Thesis is the complete solution of Conjecture 1 (in the split case) for complements of plane curves in $\mathbb{P}^2$. Our approach unifies the aforementioned methods, Corvaja-Zannier and Chen, obtaining Lang-Vojta Conjecture for complements of very generic quartics in the projective plane, which in particular completes the proof for affine varieties of log general type whose compactification is $\mathbb{P}^2$, although with a very generic hypothesis which is present neither in the original Conjecture, nor in Corvaja and Zannier's results. We used a deformation argument which reduces the general case to an extension of Corvaja and Zannier's result in the three components case. However, although the deformation argument is similar to the work of Chen, our strategy is different, making use of Logarithmic Geometry, in particular the Theory of logarithmic stable maps to Deligne-Faltings pairs, as developed by Chen and Abramovich-Chen. It turns out that this approach led to a simpler solution of the problem despite the fact that the reformulation for log-maps is more technical and requires refined tools.

I.1 What is in this Thesis

Since the problem we deal with lies in the intersection of Number Theory and Algebraic Geometry, we tried to make the exposition as self-contained as possible, assuming only basic knowledge of both fields, at the level of [Hin2] and [Har]. We hope in this way that both number theorists and algebraic geometers, can follow the parts that are more arithmetic in nature, as well as the ones that have more geometric flavor. We sketch here, for the reader's convenience, the plan of the Thesis.

The presentation is divided in three parts which can, a priori, be read separately. The first part is devoted to a throughout discussion of the origin and features of Lang-Vojta Conjecture. It contains, in particular, most of the background in Arithmetic Geometry, needed to understand the subsequent parts. At the same time it describes the motivations and the history of the Conjecture together with all the results and the ideas related to this fascinating subject. The second part is focused on the non-split case of Conjecture 1 for complements of degree four and three components divisors. Finally the Third part concerns the general case of Geometric Lang-Vojta Conjecture for complements of very generic plane curves of degree four with at most normal crossing singularities. Together with the main result it contains also an introduction to Logarithmic Geometry and to the results that we use in our proof.

We present here a more detailed description of the contents of each chapter.

The **first chapter** deals with the number field case, moving from the one dimensional classification of algebraic varieties from a Diophantine point of view, and discussing the problem for surfaces with an analysis of Bombieri-Lang Conjecture. After this first part the chapter treats the extension due to Vojta and his conjectures, with some height machinery background, together with the relationship between previous Conjecture and Vojta ones. The last section is devoted to a description of the new method developed by Corvaja and Zannier from their new proof of Siegel's Theorem and its applications to the study of integral points on surfaces.

The **second chapter** is devoted to the function field case of main Conjecture. Similarly to the first chapter it contains a discussion of the one dimensional situation related to the famous Geometric Mordell Conjecture, which serves as an introduction to function fields arithmetic and to the peculiar feature of isotriviality. Then the chapter contains a reformulation of Lang-Vojta Conjecture using the theory of integral models of algebraic varieties, which gives a unified vision of the two sides of the Conjecture. The last part of the chapter is devoted to the notion of algebraic hyperbolicity and its relationship with the main problem of this Thesis.

The **third chapter** contains a detailed discussion of known results for complements of sufficiently reducible plane curves. In particular it describes the cases of complements of four lines and of a conic and two lines. For the latter it includes a sketch of the proof as well as a description of a Theorem on Greatest Common Divisor for S-units that is used in our extension of Corvaja and Zannier's result. It ends with an overview of the generalization of previous theorems to ramified covers of $\mathbb{G}_m^2$ obtained by the same authors.

The **fourth chapter** presents the details of our proof of the result for non-split complements of degree four and three components divisors. It starts with a description of the setting, together with its similarities and differences with respect to the split case. Then it introduces the study of the moduli space for the divisors involved together with a suitable representation that is used in the proof. Finally a detailed proof with all the computation ends the chapter as well as the second part.

The **fifth chapter** deals with a comprehensive discussion of the methods present in literature for studying algebraic hyperbolicity of complements of plane curves in the algebraic geometry setting. In particular it presents the ideas and a sketch of the proof of Chen's results enlightening its power and its limitations.

The **sixth chapter** is an introduction to Logarithmic Geometry. It contains most of the definitions and the results that are needed to discuss the theory of logarithmic stable maps. Although not complete, it hopes to give the reader enough elements to enjoy the constructions as well as the strength of the results. The first section contains the basic background on logarithmic structures, logarithmic schemes and maps, together with their basic properties. The second section discusses the notion of logarithmic stable curves and its relationship with the notion of usual stable curves. The last part describes in detail the theory of logarithmic stable maps, a generalization of Kontsevich construction, focusing in particular on Chen and Abramovich-Chen theory of log stable maps to DF pairs, which is the one used in the proof of the main theorem of the third part.

The **seventh** and last **chapter** contains the proof of algebraic hyperbolicity for complements of very generic quartics. The first section is devoted to the proof of an extension of Corvaja and Zannier result for log-stable maps. Then, using this theorem, a reformulation of the problem using the moduli space of log-stable maps to DF pairs with suitable discrete data is given. Finally, using the properness of this stack, a deformation argument is described, showing that the stack is invariant under flat deformation of the quartic, and applying the aforementioned moduli interpretation, obtaining the result from the three components case.

I.2 What is *not* in this Thesis

We end this introduction with a discussion of what has been left out of this Thesis.

The first thing that we should mention here, that should have been reserved a place in our presentation, is the complex analytic analogous of Conjecture 1 and its related open problems, such as the well-known Green-Griffiths Conjecture. Actually the problem of degeneracy of integral curves in general type, or log-general type, manifolds and its relationship with the problem of distribution of rational, or integral, points is one of the starting point of Vojta's analysis. One could say that the analogies between Nevanlinna Theory and value distribution Theory and Diophantine Approximation lies at the core of Vojta's Conjectures. Even deeper, Vojta built a dictionary using which he was able to relate results in the complex analytic side with results in the Diophantine side and the other way round, like Cartan's Theorem in Nevanlinna Theory and Roth's Theorem in Diophantine approximation. This allowed him to set forth a series of striking conjectures which are only briefly described in the first Chapter, and only from the Arithmetic point of view. We refer to [Voj1] and [Voj4] for more details.

Similarly all the recent results on Hyperbolicity problems and in particular on Kobayashi Conjecture are left out from our dissertation. Since the work of Lang, a deep correlation between hyperbolicity in the sense of Kobayashi and distribution of rational and integral points on algebraic varieties has been observed and studied. This link is even more strong when one considers Diophantine problems over function fields because in this case the focus on maps from smooth projective curve into the variety has a lot of analogies with the problem of describing holomorphic maps from the complex numbers to the corresponding analytic manifold. This similarity allows one to try to adapt strategies and ideas coming from Complex Geometry in the function field case. Therefore a description of powerful techniques, such as Jet spaces and Jet differentials, and the results obtained with these tools could have been included. We decided not to include also the results about the distribution of rational curves in surfaces and higher dimensional varieties that have in common several aspects with the main problem of this Thesis. Among all the works in this direction we cite [DSW], [SY], [Dem], [McQ], [NWY], [Rou] and [DMR], which are the most related to the contents of this Thesis.

Looking at the arithmetic side of the problem considered, we decided to focus on characteristic 0 arguments, not mentioning the problems arising in characteristic p. One of the reasons was that some of the results and the techniques cannot simply be translated in this new setting. In particular some of the Conjectures are simply false if merely stated in characteristic p. As an example there are unirational surfaces of general type in positive characteristic which provide counterexamples to Bombieri-Lang Conjecture (see [AV]). For similar reasons, when considering the function field version of Conjecture 1 we almost always assume that the base field is algebraically closed, although very interesting problems emerge in the case one drops this assumption.

Finally, with the aim of keeping the exposition as self-contained as possible, we tried to limit ourselves with the use of external theorems without a brief discussion on their statements. In the cases where this has been done, we cited references where the reader can find detailed explanation of the results used and of their proofs.

I

Lang-Vojta conjecture

1

Lang-Vojta for Number Fields

Lang-Vojta Conjecture, named after Paul Vojta and Serge Lang, in its usual formulation reads as follows:

Conjecture 2 (Lang-Vojta)**.** *Let X be a quasi projective variety that can be written as $Y \setminus D$ for a (smooth) projective variety Y and an effective divisor D with normal crossing singularities, such that $K_Y + D$ is big. Then any set of S-integral points on X is not Zariski dense.*

In this chapter we are going to review the history of this Conjecture, starting from his ancestor, namely the analogous results for algebraic curves, in particular Siegel's Theorem on integral points on affine curves and Mordell Conjecture, later Faltings' Theorem. Then we will move to the two dimensional case, which is the one we are more interested in in this thesis: in particular, in the second section, we are going to review the famous Bombieri-Lang Conjecture as the first attempt to extend Faltings' Theorem to higher dimension, and to describe how Lang-Vojta Conjecture extends and generalizes it. Finally, in the last section we are going to describe new techniques, originated by the new proof of Siegel's Theorem by Corvaja and Zannier in [CZ1], which helped to partially answer Lang-Vojta Conjecture in some remarkable cases.

1.1 Integral points on Curves

The geometric classification of algebraic curves has been one of the greatest mathematical achievements of 19th century. This classification can be summarized in the following way: every complete complex curve is birational to a unique non-singular projective curve. To every curve is associated a discrete birational invariant, its genus. Finally, for a fixed genus g, isomorphism classes of projective non-singular curves of genus g form a quasi-projective variety $\mathcal{M}_g$ of dimension $\max\{3g-3, g\}$. What can be said from the arithmetic point of view? Here the problem is evidently far from being as accessible as the geometric classification. Actually the problem of describe and characterize the distribution of rational and integral points, can be considered to be completed only after Faltings proof of Mordell Conjecture and includes major theorems by, among others, Dirichlet, Mordell, Siegel, Weil and Faltings. Moreover there are still open problems regarding arithmetic features of algebraic curves.

The difference sketched between the geometric and arithmetic description of curves becomes more serious for surfaces: here a birational classification is available (although a complete description of moduli of general type surfaces is still lacking) whereas an arithmetic description of quasi-projective surfaces has barely begun and lies mainly in deep conjectures like Lang-Vojta's one.

In this section we are going to review a sort of "arithmetic classification" of curves through the most important features of Siegel and Faltings Theorems on the distribution of, respectively, integral points on affine curves and rational points on projective curves. Our main motivation, together with the beauty of these results, is to present the basis upon which the higher dimensional analogous were formulated as well as to show the interplay of different techniques which is a common aspect of Diophantine Geometry.

1.1.1 Siegel's Theorem

In [Sie] Siegel proved his celebrated Theorem on finiteness of integral points on affine curves that reads as follows

Theorem 1.1.1 (Siegel 1929,[Sie]). *Let $\mathcal{C}$ be an affine irreducible curve defined over a number field K and embedded in an affine space $\mathbb{A}^m$. Let $\tilde{\mathcal{C}}$ be the completion of its normalization and $g = g(\tilde{\mathcal{C}})$ its (geometric) genus. Let $\mathcal{O}_S$ denote the ring of S-integers of K, for a finite set S of places of K containing the archimedean ones. Suppose that either $g \geq 1$ or $g = 0$ and $\tilde{\mathcal{C}} \setminus \mathcal{C}$ contains at least three points. Then $\mathcal{C}(\mathcal{O}_S)$ is a finite set in $\mathbb{A}^m(\mathcal{O}_S)$.*

Remark 1.1.2. Historically, Siegel's original presentation of the result was split in two parts: one dealing with the case of genus zero and three points at infinity and the remaining one dealing with genus greater or equal than one. Moreover in Siegel original formulation, the set S was the set containing only the archimedean places of the number field K. The extension to number fields is due to Mahler in [Mah] for genus 1. The result has been finally extended to arbitrary finite set of places by Lang in [Lan1] using an extension of Thue-Siegel-Roth Theorem by Ridout in [Rid].

We introduce here the notion of Euler Characteristic of an affine curve which will allow us to reformulate Theorem 1.1.1 in a convenient way that will be useful in the sequel.

Definition 1.1.3. *Given an affine irreducible curve $\mathcal{C}$, let $\tilde{\mathcal{C}}$ be the completion of its normalization and let S be the set of points $\tilde{\mathcal{C}} \setminus \mathcal{C}$. We define the* Euler Characteristic *of the curve $\mathcal{C}$ to be*

$$\chi(\mathcal{C}) := \chi_S(\tilde{\mathcal{C}}) = 2g(\tilde{\mathcal{C}}) - 2 + \#S.$$

Remark 1.1.4. Classically the Euler-Poincaré Characteristic of a complete smooth curve, viewed as a compact Riemann surface, is equal to $2-2g$. In particular what we have just defined coincide with the opposite of the topological definition plus the number of points at infinity. Reasons for our choice of sign will be given in the subsequent discussion of the notion of algebraic hyperbolicity in 2.2.2.

We can now rephrase Siegel's Theorem 1.1.1 with the aid of Euler Characteristic in the following way:

Theorem 1.1.5. *Let $\mathcal{C} = \tilde{\mathcal{C}} \setminus T$ be an affine irreducible curve defined over a number field K and embedded in an affine space $\mathbb{A}^m$. Let $\mathcal{O}_S = \mathcal{O}_{S_K}$ denote the ring of S-integers of K, for a finite set S of places of K containing the archimedean ones. If $\chi(\mathcal{C}) = \chi_T(\tilde{\mathcal{C}}) > 0$ then $calC(\mathcal{O}_S)$ is a finite set in $\mathbb{A}^m(\mathcal{O}_S)$.*

Sketch of the proof. We will suppose that S is the set of the archimedean places of K; with a few modification the proof can be adapted to treat the general case.

1. Suppose we are given an infinite sequence of S-integral points (x_i) in $\mathcal{C}(\mathcal{O}_S)$. There exists a convergent subsequence (with respect to an archimedean place) in $\tilde{\mathcal{C}}$ with limit $\alpha \in \tilde{\mathcal{C}}$. In particular, being $\tilde{\mathcal{C}}(\mathcal{O}_S) \subset \tilde{\mathcal{C}}(\overline{\mathbb{Q}})$, α is algebraic.

2. If the genus is greater than zero we can embed $\tilde{\mathcal{C}}$ inside its jacobian $Jac(\mathcal{C})$: this gives a sequence in the jacobian (x_i') which is convergent, up to replacing the sequence by a subsequence. By weak Mordell-Weil Theorem the images of all the elements of a subsequence in the group $Jac(\mathcal{C})/mJac(\mathcal{C})$ coincide. Hence we can assume that $x_i' = my_i + z$ for some fixed rational point $z \in Jac(\mathcal{C})$.

3. Define the map $\phi : Jac(\mathcal{C}) \to Jac(\mathcal{C})$ by $\phi(x) = mx + z$ and put $\mathcal{D} = \phi^{-1}(\mathcal{C})$. The map is unramified and pullbacks of integral points are integral points. Hence ϕ gives us a sequence y_i of integral points that converges to some β in $\mathcal{D}$, eventually after passing to a subsequence.

4. Next compare the height function on the two curves defined by mean of two pullbacks of the same very ample divisor on the jacobian (see Definition 1.2.12 for details on Heights). Using functoriality and the properties of Neron-Tate heights on abelian varieties (in particular the quadratic behavior with respect to the isogenies on the jacobian coming from multiplication maps) we find that the height on $\mathcal{C}$ grows like (a constant times) the height on $\mathcal{D}$ to the power of $m^2/2$.

5. To conclude one applies Thue-Siegel-Roth in order to have a lower bound for the height of the x_i's on one hand and uses the above described behavior of heights together with the locally étaleness of the multiplication by m map, for m large enough, to derive a contradiction. Roughly speaking the distance between points of the sequences and their limit does not change thanks to the (local) étalness of the map ϕ; however Roth's Theorem limits the velocity by which the sequence of the heights $h(x_i)$ can converge while, on the other hand, the same speed on the cover $\mathcal{D}$ can be made arbitrarily big increasing m and this gives the contradiction to the fact that infinitely many integral points can exists on the curve $\mathcal{C}$.

6. If the genus of the curve is zero, one is led to the case of $\mathbb{P}^1$ minus, at least, three points. The conclusion in this situation comes from an application of Roth's Theorem. Equivalently one can reduce this case to the previous one in the following way: call Y this affine curve whose completion has genus 0 and which posses at least three distinct points at infinity. It can be shown (see for example Theorem 5.1, ch 5, in [FWG$^+$]) that there exists a cover Z of degree 3 of Y totally ramified over the three distinct points in the boundary of Y. Therefore, by Riemann-Hurwitz, Z is a curve of genus 1. Moreover one can show that integral points of Y lifts to integral points of the cover Z. The conclusion then follows from

the already proven result for curves of positive genus.

□

Remark 1.1.6. The above proof of Theorems 1.1.1 and 1.1.5 uses Thue-Siegel-Roth Theorem which was unavailable at the time Siegel formulated its Theorem; the original proof uses a weaker statement, namely the the Thue-Siegel Theorem. Moreover the language of jacobians was in some sense hidden in Siegel's work which relies on properties of theta characteristics. A part from these differences both the original and the modern proof follow the above described steps.

As can be seen by the previous sketch, a number of deep arithmetic and geometric theories (Heights machinery, Mordell-Weil and Thue-Siegel-Roth Theorems, the theory of Jacobians) are used throughout the proof. This is going to be a *fil rouge* crossing most of the parts of this Thesis. Another proof of this Theorem by Corvaja and Zannier in [CZ1] will be given in subsection 1.3.3 and will play a big role in the second part.

In the next subsection we are going to present Mordell Conjecture and Falting's proof. This result, stating finiteness of rational points in projective curves of genus greater than one, implies and strengthen the cases of Siegel's Theorem for those curves. Nevertheless, a part from genus zero and genus one cases, height bounds obtained for integral points may be of different type with respect to the one given for rational points coming from Faltings' Theorem; hence Theorem 1.1.1 remains of independent interest.

1.1.2 Mordell Conjecture

The history of Mordell Conjecture starts with the article [Mor1] by Mordell. In this seminal paper, after proving finite generation for the group of rational points in an elliptic curve defined over $\mathbb{Q}$, he states one of the most famous problems in Arithmetic Geometry, which has been proved by Faltings only sixty years after the original formulation. The Conjecture, now Faltings' Theorem, reads as follows

Theorem 1.1.7 (Faltings [Fal1], Mordell Conjecture)**.** *Let K be a number field and let $\mathcal{C}$ be a curve of genus greater than one defined over K. Then $\mathcal{C}(K)$ is a finite set.*

This statement was highly non-trivial and only some particular case were known at Mordell's time. Many mathematicians, although recognizing its power, were not convinced by the conjectured result. André Weil commented

> Nous sommes moins avancés à l'égard de la Conjecture de Mordell. Il s'agit là d'une question qu'un arithmeticien ne peut guère manquer de se poser; on n'aperçoit d'ailleurs aucun motif sérieux de parier pour ou contre. [Wei]

> We are less advanced in respect of the Mordell Conjecture. This is a problem that every arithmetician can hardly not ask himself; nevertheless we do see no serious reason to bet for or against its truth.

In his general audience exposition of Mordell Conjecture and Falting's ideas in [Blo], Spencer Bloch wrote «Probably most mathematicians would have agreed with Weil (certainly I would have) until [...] a German mathematician, Gerd Faltings, proved the Mordell Conjecture». This is even more revealing taking into account the proof by Grauert [Gra] and Manin [Man] (although with a gap pointed out and corrected by Coleman [Col]) of the geometric case; see section 2.1 at page 30 for further details on the function field case.

Nevertheless in 1983 Faltings presents a proof of Theorem 1.1.7 as a consequence of his proof of Tate Conjecture and Shafarevich Conjecture. His argument uses very refined and difficult tools like Arakelov Theory on moduli spaces, semistable abelian schemes and p-divisible groups. Vojta in [Voj3] (and previously for function fields in [Voj2]) gave another proof which uses ideas from classical Diophantine approximations together with technical tools of intersection theory on arithmetic threefolds developed by Gillet and Soulè. After this new proof, Faltings in [Fal2] gave another simplification, eliminating the use of Riemann-Roch Theorem for arithmetic threefolds: using his new ideas he was able to extend previous results and to prove a Conjecture of Lang. Another simplification of both Vojta and Faltings' proofs was given by Bombieri in [Bom] combining idea from Mumford [Mum] together with the ones in the aforementioned papers.

A full proof of Theorem 1.1.7 goes beyond the scope of this chapter, we refer to the following books that contains detailed and comprehensive discussion of the original proofs, together with their subsequent simplifications: Bombieri and Gubler [BG] and Hindry and Silverman [HS] discuss Bombieri's approach to Theorem 1.1.7. For an exposition of the ideas of Faltings' original paper the main source is Faltings and Wüstholz notes [FWG$^+$]; another exhaustive treatment of the original proof together with its link to Tate Conjecture and Shafarevich Conjecture can be found in Zarhin and Parshin article [ZP][(1)].

The main importance of Faltings' Theorem (together with Siegel's Theorem 1.1.1) for the purpose of this Thesis is the following corollary which completely describes the distribution of integral and rational points on algebraic curves:

Theorem 1.1.8 (Arithmetic classification of curves)**.** *Let $\mathcal{C}$ be a projective, geometrically irreducible and non-singular curve defined over a number field K. Let $\mathcal{O}_S$ be the ring of integers of K for a finite set of places containing the archimedean ones. Then, at most after a finite extension of K[(2)], the following holds:*

[(1)] For bibliographic history of this article we refer to the summary in the article's arXiv page. We just note that the paper first appeared as an appendix to the Russian version of Lang's book *Fundamentals of Diophantine Geometry*.

[(2)] This assumption is made in order to give a unified treatment of the case of genus zero and one. Indeed, at most after a finite extension, every

Table 1.1: Arithmetic classification of curves

Genus	*Rational points*	*Points at infinity*	*Integral points*
$g = 0$	*Infinite set*	≤ 2	*infinite set*
$g = 0$	*Infinite set*	≥ 3	*finite set*
$g = 1$	*Fin. generated group*	$= 0$	*infinite set*
$g = 1$	*Fin. generated group*	≥ 1	*finite set*
$g \geq 2$	*Finite set*	*Arbitrary*	*finite set*

Previous Theorem could also be restated using the Euler characteristic as defined in 1.1.3 in the same way done for Siegel's Theorem in 1.1.5. The main point of the previous classification is the fact that it exhibits a fundamental characteristic of Diophantine Geometry, namely the fact that, at least in a qualitative sense, *Geometry determines Arithmetic*. In particular for curves the genus of the complex variety associated to the arithmetic curve determines the behavior of the distribution of rational points and, together with the number of points at infinity, of integral points.

1.2 Bombieri-Lang and Lang-Vojta Conjectures

In the previous section we have seen how Siegel's Theorem 1.1.1 and Faltings' Theorem 1.1.7 completely describe the distribution of both integral and rational points on algebraic curves over number fields. In this section we will describe how these theorems for curves can, conjecturally, be extended to surfaces. The idea is that, once a suitable geometric property that extends the role of the genus for curves is determined, analogous behavior of rational points could be established. However, as the geometry of surfaces compared to the

algebraic curve possess a rational point.

geometry of curves is richer and subtler, the arithmetic of distribution of points in dimension two is far from being easy; hence an elegant and concise description as given by Theorem 1.1.8 cannot be expected for arithmetic surfaces.

The correspondent statements for surfaces are contained in the following two conjectures. The first one is due to Bombieri and Lang: Bombieri addressed the problem of degeneracy of rational points in varieties of general type in a lecture at the University of Chicago in 1980, while Lang gave more general conjectures centered on the relationship between the distribution of rational points with hyperbolicity and Diophantine approximation (see [Lan6] and [Lan2]). The conjecture reads as follows:

Conjecture 3 ((Weak) Bombieri-Lang)**.** *Let X be a surface of general type defined over a number field K. Then the set of K-rational points of X is not Zariski dense.*

While the former Conjecture can be seen as a two-dimensional analogous of Faltings' Theorem, Conjecture 2, which inspired the title of this Thesis as well as most of its results, generalized and vastly extend Siegel's Theorem on finiteness of integral points on curves with positive Euler characteristic. As mentioned at the beginning of this chapter, this Conjecture is a reformulation of the (stronger) original Conjecture by Vojta that uses ideas of Lang and admits the following analogous statement which emphasizes the link with the Bombieri-Lang Conjecture:

Conjecture 4 (Lang-Vojta II)**.** *Let X be a quasi projective variety of* log-general type *defined over a number field K and let $\mathcal{O}_S$ the ring of S-integers for a finite set of places of K containing the archimedean ones. Then the set $X(\mathcal{O}_S)$ is not Zariski dense.*

In this section we are going to describe how these two statements generalize the results of the previous section for curves, and how the geometric properties involved generalize the ones used in Siegel and Faltings Theorems. We will then describe in the detail the two conjectures and mention some of the known cases.

1.2.1 From Curves to Surfaces

From Theorem 1.1.7 one can see how the geometric properties encoded by the genus *govern* the arithmetic of the curve. Seeking a generalization to higher dimensions, and in particular to surfaces, one is led to study which geometric features of the underlying complex variety determine the distribution of rational points. The first attempt would be to study whether rational points on higher dimensional varieties are finite or not. However, parallel to this problem, one can study a different characterization of the distribution of rational points: this follows from the fact that, up to a finite extension of the base field, each rational or elliptic curve (resp. each affine curve with non positive Euler Characteristic) on the variety will carry an infinite number of rational (resp. integral) points. Therefore it is natural to consider a weaker property rather than finiteness for rational points. As a motivation consider the following

Example 1.2.1 (Corvaja and Zannier, Turchet)**.** Let $\tilde{X}$ be a smooth cubic surface defined over a number field K and let H_1, H_2 be two hyperplane sections such that $H_1 \cup H_2$ consists of 6 lines. Corvaja and Zannier in [CZ6] proved that the set of S-integral points on $X = \tilde{X} \setminus (H_1 + H_2)$ is not Zariski dense. Moreover one can prove (see [Tur]) that the only families containing infinite integral points are the 21 remaining lines in $\tilde{X}$.

This example shows how in a complement of two completely reducible hyperplane sections in a smooth cubic the S-integral points are "almost finite" in the sense that, removing a finite number of subvarieties (or a proper subvariety consisting of the union of those), the S-integral points are finite in the surface. In particular the closure of the set of integral points is a proper subvariety of the affine surface. Therefore an extension to higher dimensions of finiteness results for curves should look for non-density rather than to finiteness.

As pointed out in the arithmetic classification for curves, we should allow finitely extension of the number field of definition of the surface X. This led to the following definition:

Definition 1.2.2 (Potential Density). *Given an algebraic variety X defined over a number field K we say that the set of rational points $X(K)$ is* potentially dense *if there exists a finite extension F of K such that the set of F-rational points is Zariski dense in X.*

In order to extend Mordell and Faltings ideas to surfaces we have to look for geometric properties of algebraic surfaces who could imply that the set of rational points is not potentially dense on the surface. Therefore we need geometric properties replacing, or better extending, the role played by the genus in dimension 1. With this goal in mind we recall the following

Definition 1.2.3 (Kodaira dimension). *Let X be a smooth projective algebraic variety and let K_X be one if its canonical divisor. For each $m \geq 1$ such that the pluricanonical linear system $|mK_X|$ is not empty, i.e. such that $h^0(X, \mathcal{O}(mK_X)) \neq 0$, let*

$$\Phi_{mK_X} : X \to \mathbb{P}^N$$

be the associated map. The Kodaira dimension *of X is defined to be the number*

$$\kappa(X) = \begin{cases} -1 & \text{if } h^0(X, \mathcal{O}(mK_X)) = 0 \quad \forall m \\ \max \dim \Phi_{mK_X}(X) & \text{otherwise} . \end{cases}$$

Remark 1.2.4.

1. Some authors define $\kappa(X) = -\infty$ instead of -1 in the case in which all the pluricanonical linear systems are empty. However we prefer the previous notation because is consistent with the analogous definition of κ as the transcendent degree of the *pluricanonical ring* of X minus 1.

2. From the fact that birational morphisms between varieties make their modules of differentials being isomor-

phic follows that the Kodaira dimension of a smooth projective variety is a birational invariant.

3. It follows immediately from the definition that, for an algebraic variety X, $-1 \leq \kappa(X) \leq \dim(X)$.

We claim that the Kodaira dimension for curves gives an analogue classification like the genus one. First of all it's easy to show that for curves $\kappa = \min\{1, g-1\}$ by a case by case analysis. Let $\mathcal{C}$ define a smooth projective curve.

- When the genus of $\mathcal{C}$ is zero the canonical divisor is not effective because has degree $\deg K_{\mathcal{C}} = 2g - 2 < 0$. Hence $\kappa(\mathcal{C}) = -1$ because all the pluricanonical linear systems are empty.

- When the genus is one the canonical sheaf coincide with the structure sheaf $\mathcal{O}_{\mathcal{C}}$ for which $h^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}) = 1$ and $h^1(\mathcal{C}, \mathcal{O}_{\mathcal{C}}) = 0$. Therefore $\kappa(\mathcal{C}) = 0$.

- When the genus is greater than one the canonical divisor is ample and moreover for every $m \geq 3$ the divisor mK is very ample (see for example [HM]) and hence $\kappa(\mathcal{C}) = \dim \mathcal{C} = 1$.

Remark 1.2.5. It should be stressed that the classification of curves by their Kodaira dimension reflects the complex analytic description of Riemann Surfaces in terms of curvature: curves with negative Kodaira dimension, being of genus 0, are Riemann Surfaces with positive curvature; elliptic curves, i.e. curves with $\kappa = 0$ corresponds to flat Riemann Surfaces and finally curves with positive Kodaira dimension corresponds to negative curved Riemann Surfaces, in particular to *hyperbolic* ones.

From the previous list follows an arithmetic classification of curves based on their Kodaira dimension. In particular the following corollary of Faltings' Theorem 1.1.7 holds:

Corollary 1.2.6. *Given a smooth projective curve $\mathcal{C}$ defined over a number field K, the set of rational points $\mathcal{C}(K)$ is not po-*

tentially dense if and only if $\kappa(\mathcal{C}) = 1 = \dim \mathcal{C}$*.*

Motivated also by the previous result we recall the following terminology that extends the property of having genus greater than one for curves.

Definition 1.2.7 (General type varieties)**.** *Let* X *be an algebraic variety. If* $\kappa(X) = \dim X$ *then* X *is said to be of* general type.

For curves it is easy to see that *most* algebraic curves are of general type: in fact for almost all genus the curves are of general type and moreover the dimension of the moduli space for each genus is strictly increasing. Hence in dimension one "almost all" curves are of general type.

1.2.2 Bombieri-Lang Conjecture

The idea behind Lang and Bombieri conjectures about distribution of rational points on surfaces is that a rough analogous of the behavior exhibited by algebraic curves could hold also for surfaces. First of all we recall the Kodaira classification of surfaces which reads as follows

Theorem 1.2.8 (Kodaira Classification of Surfaces)**.** *Let* X *be an algebraic smooth surface and let* $\kappa = \kappa(X)$ *its Kodaira dimension. Then the following classification holds:*

- $\kappa = -1$*:* X *is either a Rational or a Ruled surface.*
- $\kappa = 0$*:* X *belongs to one of the following four classes: Abelian, hyperelliptic (or bi-elliptic), K3 or Enriques.*
- $\kappa = 1$*:* X *is an Elliptic Surface.*
- $\kappa = 2$*: by definition* X *is of General Type.*

Recall that there are examples of elliptic surfaces of Kodaira dimension strictly lesser than one: all Enriques surfaces and Hyperelliptic surfaces are elliptic. However each surface with $\kappa = 1$ is an elliptic surfaces. A finer classification together with a complete description of each family of surfaces not of general type follows from the *Enriques-Kodaira* Classification Theorem for surfaces (not necessarily algebraic).

We are interested in the behavior of the set of rational points for each family of surfaces listed in Theorem 1.2.8. Let us look more closer to each item appearing in the list:

- Let us consider the first case: each Rational or Ruled surface defined over a number field K is covered by rational curves which, by Theorem 1.1.7 have infinitely many rational points. Therefore for all the surfaces in this class the set of K-rational points is potentially dense.

- The case of null Kodaira dimension is more involved: it is known that rational points are potentially dense for abelian varieties and for Enriques surfaces [BT1]. There are several proved results of density of rational points for some classes of K3 surfaces [BT3] and for Hyperelliptic surfaces [BT2]. Conjectures predicts that for each of these classes the rational points *are* potentially Zariski dense.

- For elliptic surfaces of Kodaira dimension one there is a gap for an arithmetic classification of surfaces based solely on Kodaira dimension. In fact one can easily construct example of surfaces with $\kappa = 1$ that posses either a potentially dense set of rational points or a non-dense one. Consider two fibrations $X \to \mathcal{C}$ defined over a number field K having elliptic curves as fibers; suppose that the genus of the base curve $\mathcal{C}$ is greater than one: then from Faltings' Theorem 1.1.7 X has non (potentially) dense set of rational points. On the other hand if $\mathcal{C} = \mathbb{P}^1$ and there exists infinitely many sections over (a finite extension of) K, K-rational points are potentially dense in X. In both cases it may happen that X has Kodaira dimension equal to one.

- For surface of general type it is expected that the set of rational points is not potentially dense: this has been conjectured independently by Bombieri and Lang (who conjectured a more detailed description of the set of rational points and extensions also to higher dimensions). In particular, from previous discussion on Kodaira dimension for curves, it follows that Bombieri-Lang Conjecture implies Mordell Conjecture because curves of genus at least two are of general type.

Evidences for Bombieri-Lang Conjecture comes from the following Conjecture due to Lang and proved by Faltings in [Fal2] and [Fal3].

Theorem 1.2.9 (Lang Conjecture - Faltings' Big Theorem). *Let A be an abelian variety over a number field K and let X be a geometrical irreducible closed subvariety of A which is not a translate of an abelian subvariety over $\overline{K}$. Then $X \cap A(K)$ is not Zariski dense in X.*

See [Hin1] for a detailed introduction and explanation of this Conjecture. From the previous Theorem it follows a corollary which gives several evidences to Bombieri-Lang;

Corollary 1.2.10. *If X is a smooth projective variety of general type defined over a number field contained in an abelian variety, then the set of rational points of X is not Zariski dense.*

Following Noguchi's proof [Nog] in the function field case for varieties whose cotangent bundle is ample (which implies that the variety is of general type) and using Faltings' Big Theorem (cfr. Theorem 1.2.9), Moriwaki in [Mor3] obtained another evidence for Bombieri-Lang Conjecture which reads as follows:

Theorem 1.2.11 (Moriwaki). *Let X be a projective variety over a number field K. If the sheaf of differentials $\Omega^1_{X/K}$ of X over K is ample and generated by global sections, then the set of K-rational points of X is finite.*

Other evidences for Bombieri-Lang Conjecture comes from related examples and conjectures for the distribution of rational curves in general type surfaces, such as Bogomolov Theorem on the finiteness of rational and elliptic curves on general type surfaces with $c_1^2 > c_2$ [Bog].

1.2.3 Extensions and Lang-Vojta Conjecture

From the previous discussion we have seen that Conjecture 3 extends Faltings' Theorem 1.1.7. It is natural to ask whether a similar extension exists for Siegel's Theorem 1.1.1. The answer is positive and it is related to Vojta's "landmark Ph.D. Thesis", which gave the basis for a systematic treatment of analogies between Nevanlinna Theory and Diophantine Geometry over number fields. Based on this analogy Vojta formulated a set of far-reaching conjectures. For a detailed description we refer to Vojta's papers [Voj1] and [Voj4] as well as chapters in the books [HS] and [BG].

We will now recall the basic definition needed to state the main Conjecture whose specific reformulation will give Conjecture 4.

Definition 1.2.12 (Weil's Height Machinery). *Let X be a smooth projective algebraic variety defined over a number field K. There exists a (unique) map*

$$h_{X,\_} : \mathrm{Pic}(X) \to \{\textit{functions } X(\overline{K}) \to \mathbb{R}\}$$

well-defined up to bounded functions, i.e. mod $O(1)$, whose image $h_{X,D}$ for a class $D \in \mathrm{Pic}(X)$ is called a Weil height *associated to D. The map $h_{X,}$ satisfies:*

(a) *the map $D \mapsto h_{X,D}$ is an homomorphism mod $O(1)$;*

(b) *if $X = \mathbb{P}^n$ and $H \in \mathrm{Pic}(\mathbb{P}^n)$ is the class of some hyperplane in $\mathbb{P}^n$, then $h_{X,H}$ is the usual* logarithmic height *in the projective space;*

(c) *- Functoriality - for each $\overline{K}$-morphism $f : X \to Y$ of varieties and for each $D \in \mathrm{Pic}(Y)$ the following holds:*

$$h_{X,f^*D} = h_{Y,D} + O(1).$$

By abuse of notation we will indicate for a divisor $D \in \mathrm{Div}(X)$, the height corresponding to the class $\mathcal{O}(D) \in \mathrm{Pic}(X)$, with $h_{X,D}$. The previous definition can be extended to non smooth varieties (even non irreducible ones) and over any field with a set of normalized absolute values which satisfy the product formula, see [Lan6] for further details. From the previous definition one can proof the following properties for the height machinery:

Proposition 1.2.13 ([HS],[Lan6])**.** *With the above notation, the function $h_{X,}$ satisfies:*

(d) *Let D be an effective divisor in X then, up to bounded functions, $h_{X,D} \geq O(1)$;*

(e) - Northcott's Theorem *Let A be an ample divisor in X with associated height function $h_{X,A}$ then, for every constants C_1, C_2 and every extension K' of K with $[K' : K] \leq C_2$, the following set is finite*

$$\{P \in X(K') : h_{X,A}(P) \leq C_1\}.$$

The second ingredient we need to introduce to formally state Vojta Conjecture is the notion of local height. Morally we want a function which measure the v-adic distance from a point to a divisor D and such that a linear combination of this functions when v runs over the set of places gives a Weil height for the divisor D. This motivates the following

Definition 1.2.14 (Local Height)**.** *Let X be a smooth projective variety defined over a number field K. Then there exists a map*

$$\lambda\_ : \mathrm{Pic}(X) \to \{\, \textit{functions} \coprod_{v \in M_K} X \setminus \mathrm{supp} D(K_v) \to \mathbb{R}\}$$

defined up to M_K-bounded function, i.e. up to constant maps $O_v(1) : M_K \to \mathbb{R}$ that are not zero for finitely many $v \in M_K$, such that:

(a) *λ is additive up to M_K bounded functions;*

(b) *given a rational function f on X with associated divisor* $\mathrm{div}(f) = D$*. Then*

$$\lambda_{D,v}(P) = v(f(P))$$

up to $O_v(1)$, for each $v \in M_k$ where $P \in U \subset X \setminus \mathrm{supp} D(K_v)$ with U affine and $\max|P|_v = 0$ *for all but finitely many v;*

(c) *- Functoriality - for each $\overline{K}$-morphism $g : X \to Y$ of varieties and for each $D \in \mathrm{Pic}(Y)$ the following holds:*

$$\lambda_{g^*D,v} = \lambda_{D,v} \circ g + O_v(1);$$

(d) *if D is an effective divisor then $\lambda_{D,v} \geq O_v(1)$;*

(e) *if h_D is a Weyl height for D then*

$$h_D(P) = \sum_{v \in M_K} d_v \lambda_{D,v}(P) + O(1)$$

for all $P \notin \mathrm{supp} D$, with $d_v = [K_v : \mathbb{Q}_v]/[K : \mathbb{Q}]$.

For detailed construction and related properties of local height we refer to [Lan6] and [Ser]. One of the intuition behind the work of Vojta was the fact that local heights are arithmetic counterparts of proximity functions in Nevanlinna Theory: to see this consider a metrized line bundle $\mathscr{L}$ with a section s and metric $|\cdot|_v$: the function $P \mapsto \log|s(P)|_v$ is a local height at v. Following Vojta [Voj1] one can introduce arithmetic proximity and counting functions for algebraic varieties over number fields in the same spirit.

Definition 1.2.15. *Let S be a finite set of places of K, and let X, D as before. Then the following functions are well defined:*

$$m_{S,D}(P) = \sum_{v \in S} d_v \lambda_{D,v}(P)$$
$$N_{S,D}(P) = \sum_{v \notin S} d_v \lambda_{D,v}(P).$$

called the arithmetic proximity function *and* arithmetic counting function *respectively. By definition*

$$h_D(P) = N_{S,D}(P) + m_{S,D}(P).$$

With this definitions we can now state the main Vojta Conjecture which translates Griffiths' conjectural Second Main Theorem in Nevanlinna Theory.

Conjecture 5 (Vojta). *Let X be a smooth irreducible projective variety defined over a number field K and let S be a finite set of places of K. Let D be a normal crossing divisor, A an ample divisor and K_X a canonical divisor on X. Then for every $\epsilon > 0$ there exists a proper closed subset Z such that, for all $P \in X(K) \setminus Z$,*

$$m_{S,D}(P) + h_{K_X}(P) \leq \epsilon h_A(P) + O(1). \tag{1.1}$$

We end this section by two easy propositions which show how the above stated conjectured implies Bombieri-Lang Conjecture and Lang-Vojta Conjecture. We recall that one of the equivalent definition of bigness for divisor is the following: a big divisor D has a positive multiple that can be written as the sum of an ample divisor B and an effective divisor E. In the following proofs we will always assume that this multiple is the divisor itself for simplifying the notation: this can be done without loss of generality.

Proposition 1.2.16. *Vojta Conjecture 5 implies Bombieri-Lang Conjecture 3.*

Proof. If X is of general type then K_X is big, i.e. there exists a positive integer n such that $nK_X = B + E$ with B ample and E effective, and we will assume $n = 1$. Now Conjecture 5 with $D = 0$ and $A = B$ gives

$$(1 - \epsilon)h_B(P) + h_E(P) \leq O(1).$$

By Proposition 1.2.13 $h_E(P) \geq 0$ and hence, by Northcott's Theorem 1.2.13(e), the set $X(K)$ is not Zariski-dense in X. □

In order to prove that Vojta Conjecture is stronger than Lang-Vojta Conjecture we need the following reformulation of the property of being S-integral in terms of the functions defined in 1.2.15: a point P is S-integral if $N_{S,D}(P) = O(1)$ and in particular $m_{S,D}(P) = h_D(P) + O(1)$. We also recall the following:

Definition 1.2.17. *Let X be a smooth projective variety and D a normal crossing divisor on X. X is said to be of* logarithmic general type*, or log-general type, if $K_X + D$ is big for a canonical divisor K_X of X.*

Using this characterization of bigness cited before we can prove the following

Proposition 1.2.18. *Vojta Conjecture 5 implies Lang-Vojta Conjecture 4.*

Proof. For a log-general-type variety (X, D) one has

$$K_X + D = B + E,$$

for B ample and E effective. Hence Vojta Conjecture with $A = B$ gives for S-integral points

$$(1 - \epsilon)h_B(P) + h_E(P) \leq O(1).$$

Thus, using Northcott's Theorem, the set of S-integral points of (X, D) is not Zariski dense. □

1.3 New techniques from Schmidt's Subspace

In this last section of the chapter we are going to describe Schmidt's Subspace Theorem which vastly generalized Roth's Theorem. The importance of Schmidt's results in this context relies in a new proof of Siegel's Theorem 1.1.1 due to Corvaja and Zannier in [CZ1] and its implication to the study or integral points on surfaces. At the same time we will see how Schmidt's Theorem implies a particular case of Vojta Conjecture 5 for the complement in the projective space of a finite union of hyperplanes in general position.

1.3.1 Schmidt's Subspace Theorem

Wolfgang Schmidt's Theorem dealt with systems of inequalities in linear forms over $\overline{\mathbb{Q}}$ with respect to one place. Schlickewei [Sch2] and Evertse [Eve] extended the results to arbitrary set of places on a number field with better estimates of the quantities involved obtaining quantitative version of the Theorem. A stronger formulation was given in [Voj1] by Vojta. For our purposes we present the following version basically due to Schlickewei in [Sch1]:

Theorem 1.3.1 (Schmidt's Subspace Theorem)**.** *Let K be a number field, S a finite set of places, $\epsilon < 0$. For every $v \in S$ let $L_{0v}, \dots, L_{m_v v}$ be independent linear forms (in general position) in $X_1, \dots, X_n$ with coefficients in (an algebraic extension of) K. Then the set of projective solutions $x \in \mathbb{P}^n_K(K)$ of*

$$\prod_{v \in S} \prod_{i=0}^{m_v} \frac{|L_{iv}(x)|_v}{|x|_v} < H(x)^{-n-1-\epsilon}$$

is contained in the union $T_1 \cup \cdots \cup T_h$ for finitely many hyperplanes $T_1, \dots, T_h$ in $\mathbb{P}^n_K$. Here $H(\cdot)$ is the multiplicative projective height $H(x_1 : \cdots : x_d) = \prod_v \max(|x_1|_v, \dots, |x_d|_v)$.

Remark 1.3.2. It can easily be seen that Theorem 1.3.1 extends Roth's Theorem: it is sufficient to consider the case for the infinite place ∞ of $\mathbb{Q}$ in which $m_\infty = n = 2$, $L_1(x,y) = x - \alpha y$, and $L_2(x,y) = y$.

Theorem 1.3.1 has several application in Diophantine approximation of algebraic numbers which cannot fit in this short analysis of its statement; we refer the reader to detailed description in [BG] and [Bil].

1.3.2 Lang-Vojta for many components

Schmidt's subspace Theorem can be used to prove Vojta Conjecture 5 for the complement of hyperplanes in general position in the projective space.

Theorem 1.3.3. *Conjecture 5 holds in $\mathbb{P}^n$ for D equal to the sum $H_1 + \cdots + H_m$ where H_i is an hyperplane and $\{H_i\}$ are in general position.*

Proof. Without loss of generality we can assume that each hyperplane is defined over the number field K, at most enlarging K and considering a base change. Now the local height for $\mathbb{P}^n$ with respect to the hyperplane L_i is

$$-\log \frac{|L_i(x)|_v}{|x|_v},$$

where $|x|_v := \max_j |x_j|_v$. Therefore the proximity function for D can be written as

$$m_{S,D}(x) = -\sum_{v \in S} \sum_i \log \left(\frac{|L_i(x)|_v}{|x|_v} \right) + O(1).$$

On the other hand $K_{\mathbb{P}^n} = \mathcal{O}_{\mathbb{P}^n}(-n-1)$ and hence

$$h_{K_{\mathbb{P}^n}} = -(n+1)h + O(1).$$

Vojta Conjecture then has the following form:

$$-\sum_{v \in S} \sum_i \log \left(\frac{|L_i(x)|_v}{|x|_v} \right) - (n+1)h(x) \leq \epsilon h(x) + O(1),$$

which is equivalent to (the logarithmic version of) Schmidt's Subspace Theorem. □

In particular, as we are mostly interested in application to two dimensional varieties, an easy corollary is the following:

Corollary 1.3.4. *Given 4 lines $D_1, \dots, D_4$ in $\mathbb{P}^2_K$ defined over a number field K and S a finite set of places containing the archimedean ones, the set of S-integral points on the complement of $D_1 + \cdots + D_4$ is not Zariski dense.*

Notice that the divisor formed by four lines in general position, makes the complement $\mathbb{P}^2 \setminus D$ where $D = D_1 + \dots D_4$

a variety of log-general type, because $K_{\mathbb{P}^2} + D \sim \mathcal{O}_{\mathbb{P}^2}(1)$ is an ample divisor. In particular Lang-Vojta Conjecture 4 holds for the complement of at least $n+2$ hyperplanes in general position in $\mathbb{P}^n$.

1.3.3 A new proof of Siegel's Theorem

In [CZ1] Corvaja and Zannier gave another proof of Theorem 1.1.1 avoiding the embedding in the jacobian and replacing the use of Roth's Theorem in the original proof with the stronger Subspace Theorem (cfr. Theorem 1.3.1). The importance of this new reformulation, aside from simplifying Siegel's argument, relies on extension to higher dimension which will be crucial in next parts of this Thesis.

Let us briefly summarize the ideas behind Corvaja and Zannier work. The statement we are going to recall is slightly weaker than Theorem 1.1.1 nevertheless a covering argument gives the full conclusion of Siegel's Theorem as a corollary of the following

Theorem 1.3.5 (Siegel, Corvaja and Zannier)**.** *Let $\tilde{C}$ be an irreducible projective smooth curve defined over a number field K and let C be a non-empty affine subset of $\tilde{C}$ embedded in an affine space $\mathbb{A}^m$. Assume that $\#(\tilde{C} \setminus C) \geq 3$, then for a finite set of places S containing the infinite ones, the set $C(\mathcal{O}_S)$ is finite.*

We briefly sketch the proof of this Theorem in order to describe the main ideas.

Sketch of the proof. Let D be the divisor in $\tilde{C}$ whose irreducible components are the points at infinity $\tilde{C} \setminus C$ and suppose that D is supported on r points. For $n \gg 0$ one has

$$h^0(\tilde{C}, nD) = nr + O(1) =: l.$$

Let $\phi_1, \ldots, \phi_l$ be a basis for $\mathcal{O}(D)$: we can assume that for each S-integral point P, $\phi_j(P) \in \mathcal{O}_S$. This follows from the fact that, up to multiplication by a non-zero constant, each ϕ_j is integral over $\mathcal{O}_S[x_1, \ldots, x_n]$, where $x_1, \ldots, x_n$ are coordinate functions on $\mathcal{C}$. Assume now that there exist infinitely many distinct integral points $\{P_i\}$. Up to passing to a subsequence, we may assume that for each $v \in S$, the P_i converge v-adically to a point $Q_v \in \tilde{\mathcal{C}}(K_v)$.

Estimate for $|\phi_j(P_i)|$ are the following:

$$\|\Phi(P_i)\| \ll \begin{cases} |t_v(P_i)|_v^{-n}, & \text{if } Q_v \in \tilde{\mathcal{C}} \setminus \mathcal{C} \\ 1, & \text{otherwise} \end{cases}$$

where $\Phi = (\phi_1, \ldots, \phi_l)$ and t_v is the local parameter at Q_v. Taking into account the fact that $\phi_j(P_i)$ are S-integers, and putting $S' = \{v \in M_K : Q_v \in \tilde{\mathcal{C}} \setminus \mathcal{C}\}$ one obtains that

$$H(\Phi(P_i)) = \prod_{v \in S} \|\Phi(P_i)\| \ll \prod_{V \in S'} |t_v(P_i)|_v^{-n}.$$

The idea is now to consider elements z of $\mathcal{O}(nD)$ with specified vanishing order at Q_v: for this functions $|z(P_i)|_v$ becomes "small" as P_i approaches Q_v. Now the vector space of $z \in \mathcal{O}(nD)$ with prescribed order k of zero at Q_v can be choose to have positive dimension for specific values of k. Every z can moreover be written as a linear form in terms of the ϕ_i. Choosing appropriately vector spaces varying k gives rise to linear independent linear forms in the ϕ_i which verify an equations as in Schmidt's Subspace Theorem. Then the affine version of Theorem 1.3.1 can be applied and one finds out that only finitely many among the P_i are distinct which contradicts the original assumption. □

This new proof of Siegel's Theorem has the advantage to be suitable for higher dimensional extension. In particular the same authors obtained a number of strong results on degeneracy of integral points on surfaces by means of this new strategy. Among these results we cite the following which extends Theorem 1.3.3 to more general situations.

Theorem 1.3.6 ([CZ4]). *Let X be a geometrically irreducible nonsingular projective surface defined over a number field K and let $D_1, \dots, D_4$ be irreducible effective divisors such that*

1. *No three of them shares a common point;*
2. *For all $i \neq j$, $\operatorname{supp} D_i \cap \operatorname{supp} D_j \neq \emptyset$;*
3. *For all $i \neq j$, $D_i \underset{num}{\sim} mD_j$ for a certain $m = m_{i,j} \in \mathbb{Z}$.*

Then no set of S-integral points in $X \setminus D$ is Zariski dense.

Remark 1.3.7.

- Clearly for $X = \mathbb{P}^2$ all the hypotheses of the Theorem are verified for the divisor D consisting of four lines in general position and hence the Theorem implies Corollary 1.3.4. In particular each $D_i = L_i$ is effective (even ample in this case), the general position hypothesis implies condition 1 and 2, while condition 3 follows from the fact that, being $\operatorname{Pic}\mathbb{P}^2$ of rank 1, all classes of lines are linearly equivalent.
- The proof of the previous Theorem relies on a generalization of the ideas of Theorem 1.3.5 where a suitable choice of a linear system of multiples of the irreducible components D_i replaces the linear spaces of rational functions with prescribed order of zeros at limit points.

A more general theorem can be found in [CZ2] and, with some modification and extension, in [CZ4]. It also worth mentioning a Corollary obtained by Levin in [Lev] where he was able to drop the third hypothesis on the D_i provided that D_i is ample for every i.

2

Function Fields

Function fields in one variable and Number fields share several properties. This deep analogy was observed from the second half of the 19th century; one of the first systematic treatment can be found in the famous paper by Dedekind and Weber [DW]. Further descriptions, due to Kronecker, Weil and van der Waerden, settled this profound connection which finally became formally completed with the scheme theory developed by Grothendieck. In this chapter we are going to discuss how this analogy can be carried over Diophantine problems. In the first section we describe the analogous of Mordell Conjecture over function fields, which later became Manin's (although with a gap) and, independently, Grauert's Theorem in [Man] and [Gra] respectively. This serves both as an introduction to arithmetic over function field and as a starting point for the generalization to higher dimension that led to the formulation of Lang-Vojta Conjecture over function fields. After recalling the notion of model which permits to give a unified definition for rational and integral points for varieties defined both over number fields and over function fields, in the last section we formulate Lang-Vojta Conjecture for function fields, both in the split and the non split case, and we connect it with the notion of algebraic hyperbolicity.

2.1 Geometric Mordell

In [Lan1] Serge Lang stated the following conjecture, which has become known as Geometric Mordell Conjecture over $\mathbb{C}$:

Conjecture 6 (Geometric Mordell). *Let $\mathcal{C}$ be a curve of genus greater than 1 defined over a (complex) function field K of an algebraic regular curve $\mathcal{B}$, viewed as fibered surface $\pi : \mathcal{C} \to \mathcal{B}$. If the generic fiber $\mathcal{C}_t$ has infinitely many rational points in K then there exists a covering $\mathcal{B}' \to \mathcal{B}$ such that the base change $\mathcal{C} \times_K K'$, where K' is the function field of $\mathcal{B}'$, is isomorphic over K' to $\mathcal{C}_0 \times_{\mathbb{C}} K'$ for a curve $\mathcal{C}_0$ defined over $\mathbb{C}$. Moreover all but a finite number of these sections arise from constant points of the fixed curve.*

This Conjecture is obtained from Mordell Conjecture 1.1.7 replacing the number field with the a function field (in one variable). However a bare translation could not work as we are going to see in subsection2.1.2 and the isotriviality question should be addressed. In order to properly describe the features of the previous Conjecture as well as the ideas behind its proof by Grauert [Gra] we need first a precise extension of arithmetic notions over number fields into the framework of function field arithmetic.

2.1.1 Function fields and Number fields

As pointed out in the preface to this chapter function fields in one variable behave very similarly to number fields. We start this analogy with the following

Definition 2.1.1 (Function Field). *A* Function Field *K over an algebraically closed field k is a finitely generated field extension of finite transcendence degree over k. A function field* in one variable, *or equivalently a function field of an algebraic curve, is a function field with transcendence degree equal to one, i.e. a field K/k for which it exists an element x, transcendental over k, such that K is a finite extension of $k(x)$.*

Remark 2.1.2. With the language of schemes the function field of a curve X, or more general of every integral scheme over an algebraic closed field, can be recovered form the structure sheaf $\mathcal{O}_X$ in the following way: given any affine open subset of X the function field of X is the fraction field of $\mathcal{O}_X(V)$. Moreover, if η is the (unique) generic point of X, the function field of X is also isomorphic to the stalk $\mathcal{O}_{X,\eta}$.

The analogy between number fields and function fields of curves, also known as algebraic function fields in one variable, comes from the fact that one-dimensional affine integral regular schemes are either smooth affine curves over a field k or open subset of the spectrum of the ring of integers of a number field. Formally, given a number field k with ring of integers $\mathcal{O}$ the scheme $Spec\,\mathcal{O}$ is one dimensional affine and integral. From this analogy, that was already present before the introduction of Grothendieck schemes' theory, several classical properties of number fields find an analogue in the theory of function field. In particular the theory of the heights can be defined over function fields.

Definition 2.1.3. *Given a function field K in one variable of a non singular curve $\mathcal{C}$, each (geometric) point $P \in \mathcal{C}$ determines a non trivial absolute value by*

$$|f|_P := e^{-\mathrm{ord}_P(f)}.$$

Moreover if $Q \neq P$ then the absolute values $|\cdot|_Q$ and $|\cdot|_P$ are not equivalent.

Remark 2.1.4.

- The definition could have been given more generally for function fields of algebraic varieties regular in codimension one (or rather for regular models of higher dimensional function fields), replacing the point P with prime divisors. Extensions exist also for function fields over non-algebraically closed fields in which one should replace points with orbits under the absolute Galois group.
- From the fact that any rational function f on a projec-

tive curve has an associated divisor of degree zero, it follows that the set of absolute values satisfy the product formula.

Given the set of absolute values M_K for a function field in one variable K, normalized in such a way that they satisfy the product formula, heights can be defined for K in the following way:

Definition 2.1.5. *Let $K = K(\mathcal{C})$ be as before. For any $f \in K$ the* height *of f is*

$$h(f) = -\sum_{P \in \mathcal{C}} \min\{0, \mathrm{ord}_P(f)\} = \sum_{P \in \mathcal{C}} \max\{0, \mathrm{ord}_P(f)\}.$$

In the same way for a point $g \in \mathbb{P}^n(K)$, $g = (f_0 : \cdots : f_n)$, its height is defined as

$$h(g) = -\sum_{P \in \mathcal{C}} \min_i \{\mathrm{ord}_P(f_i)\}.$$

From the definition it follows that a rational function on a regular curve has no poles if and only if its height is zero if and only if it is constant on the curve.

We end this subsection with a table illustrating the interplay and the similarity between number fields and function fields. We stress in particular how each geometric object in the right column, in particular dominant maps and pullbacks, are analogous of purely arithmetic notions like extensions of fields and extensions of ideals. This analogy can be brought further using Arakelov Theory and extending the notion of divisors to number fields by suitably compactifying the affine curve $Spec\, \mathcal{O}_S$; in this framework an intersection theory can be defined for such compactified divisors sharing many analogous properties of intersection theory in the geometric side. We refer to [Lan5] for further details on this subject.

Table 2.1: Number Fields and Function Fields analogy

Number Field	Function Field
$\mathbb{Z}$	$k[x]$
$\mathbb{Q}$	$k(x)$
$\mathbb{Q}_p$	$k((x))$
K finite extension of $\mathbb{Q}$	K function field of $\mathcal{C}$
place	geometric point
finite set of places	finite set of points
ring of S-integers	ring of regular functions
$Spec\ \mathcal{O}_{K,S}$	Affine curve $\mathcal{C} \setminus S$
product formula	deg principal divisor = 0
extension of number fields	dominant maps
extension of ideals	pullback of divisors

2.1.2 Mordell Conjecture for function fields

Given the fact that many similarity between number and function fields exist, it is natural to ask if theorems and conjectures about arithmetic (in a broader sense) still hold when replacing the occurrences of "number field" with "function field in one variable" with proper modification according to Table 2.1. Thus it seems reasonable to ask this question for Mordell Conjecture, as done by Lang. However, as the following example shows, we cannot expect Faltings' Theorem 1.1.7 to hold without a careful analysis of the new situation.

Example 2.1.6. Let $\mathcal{C}$ be a curve of genus greater than one defined over $\mathbb{C}$ and consider the trivial family $\mathcal{C} \times \mathbb{P}^1 \to \mathbb{P}^1$. The family can be viewed as the curve $\mathcal{C}$ (trivially) defined over the function field $K(t)$ of $\mathbb{P}^1$. All the fibers of the family, being isomorphic to $\mathcal{C}$ have genus greater than one. Now Mordell Conjecture over function fields, without any other restriction, should imply that the set of $\mathbb{C}(t)$-rational points of $\mathcal{C}$, i.e. section $\mathbb{P}^1 \to \mathcal{C} \times \mathbb{P}^1$, are finite. However this is easily seen to be false by considering constant sections $\mathbb{P}^1 \to \{P\} \times \mathbb{P}^1$ for each point $P \in \mathcal{C}$. In particular the general type curve $\mathcal{C}$ defined over $C(t)$ has infinitely many $\mathbb{C}(t)$-rational points.

From the previous example one could guess that the problem relied on the fact that the family was a product and the curve $\mathcal{C}$ was actually defined over the base field $\mathbb{C}$ rather than on the function field $\mathbb{C}(t)$, i.e. the family was trivial. However, as the following example shows, things can go wrong even for non trivial families.

Example 2.1.7 (Gasbarri [Gas]). Consider the curve $\mathcal{C} := (x + ty)^4 + y^4$ defined over $\mathbb{C}(t)$. It has an associated fibration $\mathcal{C} \to \mathbb{P}^1$ whose generic fiber $\mathcal{C}_{t_0} = (x + t_0 y)^4 + y^4$ is a smooth projective curve of genus 3. Again if we consider the same statement of 1.1.7 only replacing the number field with the function field $\mathbb{C}(t)$ we would expect that the number of $\mathbb{C}(t)$-rational points of $\mathcal{C}$ to be finite. However we claim that $\mathcal{C}(\mathbb{C}(t))$ is infinite; to see this consider the equation $\alpha^4 + \beta^4 = 1$ over $\mathbb{C}^2$: it has infinitely many solutions. Each solution gives a $\mathbb{C}$-point of $\mathcal{C}_{t_0}$, namely $(\alpha - t_0\beta, \beta)$ proving the claim. Moreover the family is not trivial in the sense of the previous example, i.e. $\mathcal{C}$ is not defined over $\mathbb{C}$. Notice however that each fiber of the family is isomorphic to the curve $x^4 + y^4 = 1$ via $x + ty \mapsto x$ and $y \mapsto y$.

Motivated by the previous examples we give the following:

Definition 2.1.8. *Given a family of irreducible, smooth projective curves $\mathcal{C} \to \mathcal{B}$ over a smooth base $\mathcal{B}$, we say that the family is* isotrivial *if all the fibers $\mathcal{C}_b$ are isomorphic to a fixed curve $\mathcal{C}_0$. With abuse of notation we will say that a curve $\mathcal{C}$ defined over a function field K is* isotrivial *if the corresponding fibration $\mathcal{C} \to \mathcal{B}$ is isotrivial, where $\mathcal{B}$ is a curve with function field K.*

Isotriviality extends the notion of (birational) triviality for family of curves, i.e. a product of curves fibered over one of the factors is immediately isotrivial. At the same time this notion encompass many other families that are not products, like the one defined in the previous example. However, after a cover of the base of the family, each isotrivial family becomes trivial; in particular the following easy lemma holds:

Lemma 2.1.9. *Given a isotrivial family $\mathcal{C} \to \mathcal{B}$ of smooth projective irreducible curves, there exists a cover $\mathcal{B}' \to \mathcal{B}$ such that the base changed family $\mathcal{C} \times_{\mathcal{B}} \mathcal{B}' \to \mathcal{B}'$ with fibration given by second projection, is a trivial family, i.e. is isomorphic to a product $\mathcal{C} \times_{\mathbb{C}} \mathcal{B}_0$.*

The Lemma implies that rational points for curves defined over function fields will always be not finite for isotrivial curves, at most after a base change. The analogous of Mordell Conjecture for function fields thus asks whether this holds only for this class of curves. We can then restate Conjecture 6 in the following way:

Conjecture 7. *Let $\mathcal{C}$ be a smooth projective curve defined over a function field K of genus greater than 1. If $\mathcal{C}(K)$ is infinite then $\mathcal{C}$ is isotrivial.*

The Conjecture has been proved in the sixties by Manin [Man] (although with a gap fixed by Coleman [Col]) using analytic arguments and later by Grauert [Gra] using algebraic methods. Samuel in [Sam1] gave a proof in characteristic p using ideas of Grauert. A detailed explanation of Grauert methods can be found in Samuel's survey [Sam2]. In Mazur's detailed discussion of Faltings' proof of Mordell Conjecture [Maz] he stress the role of Arakelov [Ara] and Zahrin [Zar] results which imply new proofs of Geometric Mordell, i.e. Mordell Conjecture over function fields, using ideas of Parshin: this gives, if necessary, even more importance to the geometric case.

One of the ideas of Grauert's proof which is central in some of the higher dimensional extensions is the following: suppose $\mathcal{C}$ is a curve defined over a function field K of a curve $\mathcal{B}$, corresponding to a fibration $\pi : X \to \mathcal{B}$. Then one can prove that almost all sections of the fibration, which correspond to rational points, verify a first order differential equation, i.e. almost all sections are tangent to a given horizontal vector field. Formally each section $\sigma : \mathcal{B} \to X$ can be lifted to the projective bundle $\mathcal{B} \to \mathbb{P}(\Omega^1_X) = Proj(\mathrm{Sym}(\Omega^1_X))$ via the surjective map $\sigma^*\Omega^1_X \to \Omega^1_{\mathcal{B}}$. Grauert proves (in a different language) that there exists a section ϕ of a suitable line bundle over $\mathbb{P}(\Omega^1)$ whose zero section contains all but finitely many images of sections. He then concludes that if infinitely many sections exist, given the fact that they satisfy the differential equation given by $\phi = 0$, a splitting is provided for the relative tangent sequence which implies that the family is isotrivial (via the vanishing of the Kodaira-Spencer class).

In particular Grauert's construction gives first insights for the theory of Jet spaces which plays a role in some degeneracy result for complex analytic analogues. In this direction recent analogue of Conjecture 7 in higher dimension have been proved by Mourougane [Mou] for very general hypersurfaces in the projective space of high enough degree using proper extension of the ideas briefly described above.

2.2 Geometric Lang-Vojta

The previous discussion about similarity between number fields and function fields can be brought further encompassing the notion of integral points. Moreover, another formulation of Vojta-Lang Conjecture can be given from a scheme-theoretic point of view via the notion of *model*. In particular, using insights coming from Conjecture 6 we can restate Lang-Vojta Conjecture for number field case and its natural extensions to function fields in a unified way as follows.

2.2.1 Integral models and Lang-Vojta Conjecture

Let K be a number field, let $\mathcal{O}_K$ denote the ring of integers of K and let $\mathcal{O}_S$ denote the ring of S-integers for a finite set of places S containing the infinite ones. We give the following

Definition 2.2.1. *Let X be a smooth projective variety defined over the number field K. A (integral)* model *for X is a projective variety $\mathcal{X}$ together with a flat morphism $\mathcal{X} \to Spec\, \mathcal{O}_K$ such that*

$$X \simeq Spec\, K \times_{Spec\, \mathcal{O}_K} \mathcal{X}.$$

Informally we can think $\mathcal{X}$ as a fibration over $Spec\, \mathcal{O}_K$ whose central fiber over $Spec\, K$ is (isomorphic to) X.

The definition of model can be generalized to deal with affine varieties in a standard way. Moreover, if K is the function field of a smooth integral projective curve $\mathcal{C}$, one can define a *model* for the variety X over $K = k(\mathcal{C})$ replacing $Spec\, \mathcal{O}_K$ with $\mathcal{C}$. In this way one can also see better the fibration interpretation given before.

We are interested in describing rational and integral points for varieties defined either over number fields or function fields. For rational points, i.e. points of $X(K)$, the following fact holds: let X be as before and fix an integral model $\mathcal{X} \to B$ with $B = Spec\, \mathcal{O}_K$ for number fields, and $B = \mathcal{C}$ for function fields, then

Rational Points $X(K)$ correspond bijectively to sections $\pi : B \to \mathcal{X}$

A similar description holds for integral points. Let $\tilde{\mathcal{X}}$ be a smooth projective variety and $D \in \mathrm{Div}(\tilde{\mathcal{X}})$ a divisor with normal crossing singularities both defined over the number field K; choose a model $\tilde{\mathcal{X}}, \mathcal{D}$ for $\tilde{\mathcal{X}}$ and D over $\mathcal{O}_S$: in the same way that rational points correspond to sections

$$\pi_x : Spec\, (\mathcal{O}_S) \to \mathcal{X},$$

S-integral points (or rather D, S-integral points) correspond to sections such that π_x does not intersect $\mathcal{D}$ over points $v \in Spec\, (\mathcal{O}_S) \setminus S$, or equivalently such that $\pi^* D$ is supported over S. As before one can replace the number field K with a function field of a smooth integral projective curve $\mathcal{C}$ and $Spec\,(\mathcal{O}_S)$ with $\mathcal{C} \setminus S$ for a finite set of points in $\mathcal{C}$. Thus we can reformulate this, denoting by B either $Spec\,(\mathcal{O}_S)$ or $\mathcal{C} \setminus S$, saying that

$$\boxed{\begin{array}{c}(S,\mathcal{D})\text{-integral points}\\ \text{corresponds bijectively to}\\ \text{sections } \pi : B \to \mathcal{X} \text{ such that}\\ supp(\pi^*\mathcal{D}) \subset S.\end{array}}$$

Note that when the field K is replaced by a function field of a curve $K = k(\tilde{\mathcal{C}})$ (assume that the characteristic of the base field k is zero, although this is not necessary for the statement of the Conjecture) a structure projection $\pi : \tilde{\mathcal{X}} \to \tilde{\mathcal{C}}$ is naturally given by the model, where the generic fiber X_t will be isomorphic to $\mathcal{X} = \tilde{X} \setminus D$. Nevertheless, as in Section 2.1, a distinction should be made in order to deal with trivial or isotrivial families. We gave the following definition for the two dimensional case:

Definition 2.2.2. *Let X be an affine surface embedded in an algebraic projective smooth variety $\tilde{X}$ as $X \cong \tilde{X} \setminus D$ for a normal crossing divisor D, where $X, \tilde{X}$ and D are defined over a function field $K = k(\tilde{\mathcal{C}})$ corresponding to an affine fibration $\mathcal{X} \to \mathcal{C}$. The fibration $\mathcal{X} \to \mathcal{C}$ is called* split *if $\mathcal{X} \cong \mathcal{X}_k \otimes_k \mathcal{C}$ over k, i.e. the fibration is trivial. In particular we call X split if the affine variety $\mathcal{X}$ is defined over k.*

Remark 2.2.3. One may argue that the previous definition defines birational trivial affine fibration which is certainly true but we will see that a major role will be played by these fibrations in the sequel. In particular in the next section we are going to define a different version of Conjecture 2 for function fields in which the main property will be independent by base change of the original variety. In particular this would imply that each isotrivial variety could be consider as birationally trivial and hence “split".

For a good introduction to the theory of models over function fields and isotriviality problem see [Gas] sections 1 to 3.

The original formulation of Conjecture 2 for function fields reads as follows:

Conjecture 8. *Let X be an affine algebraic surface defined over a function field $K = k(\tilde{C})$ for a smooth projective curve $\tilde{C}$ and such that the generic point X_t is of log-general type. If there exist sections of the structure projection $\pi : \mathcal{X} \to \tilde{C}$ of arbitrarily large height and such that the union of their images is dense then X is isotrivial.*

Notice the strong analogy with the previous statement and Conjecture 7: in particular the case in which the X has dimension one, the divisor is empty, i.e. X is of general type or equivalently of genus greater than one, is precisely Geometric Mordell. Therefore some evidences for Conjecture 8 comes for the proofs of Geometric Mordell as discussed in subsection 2.1.2. Moreover, by the previous unified description of rational and integral points over both number fields and function fields, one can see how morphisms $\tilde{C} \to X$ will play an important role in the treatment of the previous Conjecture.

2.2.2 Algebraic Hyperbolicity

Conjecture 8 will not be the focus of the following parts of this Thesis but rather another related formulation slightly weaker that comes from the following observation made in the holomorphic curves setting. Recall that a complex compact manifold is said to be (Brody) *hyperbolic* if every holomorphic map from $\mathbb{C}$ to the manifold is constant; in the compact case Brody hyperbolicity is equivalent to Kobayashi hyperbolicity, equivalence following from Brody's Lemma (see for details [Bro]). In particular, for algebraic manifolds, hyperbolicity would imply that all holomorphic maps from abelian varieties must be constant. Kobayashi [Kob] and Lang [Lan4] made conjectures about the relationship between hyperbolicity and algebraic properties of the manifolds and whether such properties would imply, or being equivalent to, hyperbolicity. One consequence of hyperbolicity for compact manifolds is the following:

Theorem 2.2.4 (Demailly [Dem]). *Let X be a projective complex variety immersed in some projective space for a choice of a very ample line bundle. Then if the associated manifold is hyperbolic the following holds: there exists a constant $A > 0$ such that each irreducible curve $\mathcal{C} \subset X$ satisfies*

$$\deg \mathcal{C} \leq A(2g(\tilde{\mathcal{C}}) - 2) = A\chi(\tilde{\mathcal{C}}),$$

where $\tilde{\mathcal{C}}$ is the normalization of $\mathcal{C}$.

Following this Theorem, Demailly introduced the following notion which will be one of the focuses of this Thesis:

Definition 2.2.5. *A smooth projective variety X is* algebraically hyperbolic *if there exists a constant A such that for each irreducible curve $\mathcal{C} \subset X$ the following holds:*

$$\deg \mathcal{C} \leq A\chi(\tilde{\mathcal{C}}).$$

Using strong analogies between hyperbolicity and degeneracy of rational points Lang conjectured that a general type variety should be hyperbolic and in particular algebraically hyperbolic. This has been extended to the relative case giving the following

Conjecture 9. *Given an affine variety X embedded as $\tilde{X} \setminus D$ for a smooth projective variety $\tilde{X}$ and a normal crossing divisor D, if X is of log-general type then there exists a proper subvariety Exc (called the* exceptional set*) such that there exists a bound for the degree of images of non-constant morphisms $\mathcal{C} \to X$ from affine curves whose image is not entire contained in Exc, in terms of the Euler Characteristic of $\mathcal{C}$.*

Euler Characteristic for affine abstract curves was defined in 1.1.3 at page 5. We stress that Conjecture 9 is a special case of Conjecture 8 because maps $\mathcal{C} \to X$ from an affine curve with normalization $\tilde{\mathcal{C}} \setminus S$ correspond to (S, D)-integral points for X as defined over the function field of the curve $\mathcal{C}$. Now a bound for the degree of their images gives an height bound for such integral points and therefore implies that they cannot be Zariski dense in X unless X is not isotrivial. Finally we can now state the conjecture we are going to devote most of this Thesis, namely a weaker version of Conjecture 9: the conjecture deals with logarithmic general type affine varieties in which there could exists curves with negative Euler characteristic. This implies that in general such varieties will not be algebraically hyperbolic. Nevertheless a slight modification of the bound of Theorem 2.2.4 it is expected to hold.

Conjecture 10. *Given $X, \tilde{X}, D$ as before, if X is of log general type then there exists a constant A such that for each affine curve $\mathcal{C}$ with normalization $\tilde{\mathcal{C}} \setminus S$ for a smooth projective curve $\tilde{\mathcal{C}}$, a finite set of points $S \subset \tilde{\mathcal{C}}$ and for all non-constant morphisms $\varphi : \mathcal{C} \to X$, the following holds:*

$$\deg \varphi(\tilde{\mathcal{C}}) \leq A \max\{1, \chi_S(\mathcal{C})\}.$$

Conjecture 10 is slightly weaker than Conjecture 9 because it allows maps from curves of non-negative Euler Characteristic to X. However, if images of curves with negative Euler characteristic are all contained in a proper subvariety of X, e.g. there exist only finitely many images of such curves in X, this implies the conclusion of Conjecture 9. For the property described in the previous Conjecture, being a weaker version of algebraic hyperbolicity, one can give the following

Definition 2.2.6 (Weak Algebraic Hyberbolicity)**.** *An affine variety X embedded as $\tilde{X} \setminus D$ in a smooth projective variety $\tilde{X}$, for a normal crossing divisor $D \in \operatorname{Pic}\tilde{X}$, is called* weakly algebraically hyperbolic *if there exists a constant A such that for each irreducible affine curve $\mathcal{C} \subset X$ the following holds*

$$\deg \tilde{\mathcal{C}} \leq A \max\{1, \chi(\mathcal{C})\}.$$

From now on, when we refer to algebraic hyperbolicity we will always considering *weak* algebraic hyperbolicity. When we will need to refer to the usual algebraic hyperbolicity we will explicitly stress the difference, or we will call it *strong* algebraic hyperbolicity.

Remark 2.2.7. Algebraic Hyperbolicity in its weaker form seems to be the natural generalization of the projective case to the affine case. If one considers the simplest case of complements of a normal crossing divisor D in $\mathbb{P}^2$ one sees that the smallest degree D can have in order for $\mathbb{P}^2 \setminus D$ to be of log-general type is 4. However, each quartic possess 28 bi-tangent lines (Plücker Theorem) and to each of these lines one can define a map

$$\mathbb{G}_m \to \mathbb{P}^2 \setminus D$$

given by the restriction of the map $\mathbb{P}^1 \to \mathbb{P}^2$ whose image is a bi-tangent. Hence $\mathbb{P}^2 \setminus D$ always admits maps from affine curves with non positive Euler Characteristic and hence is *not* strongly algebraically hyperbolic while it is expected to be weakly algebraically hyperbolic. Nevertheless if these 28 maps, together with the 24 associated to the tangents to the flexes, are the only $\mathbb{G}_m$ immersion in $\mathbb{P}^2 \setminus D$ this would not violate Conjecture 9.

II

Three components case

3

Known results

Before moving to our first original result of this Thesis we are going to review in detail what is known about Lang-Vojta Conjecture for function field when the compactification of the affine variety is the projective plane $\mathbb{P}^2$. First of all we are going to describe the classical techniques available when the divisor at infinity has several irreducible components focusing in particular on the case of the complement of four lines. This presentation will allow us to see how the ideas of Corvaja and Zannier supersede the previous approach leading to the solution for complements of a conic and two lines. We will describe their approach and the greatest common divisor formulation which will be essential in the next chapter. Finally we will briefly describe the ideas of the same authors for a generalization to other type of surface which implies the results for $\mathbb{P}^2$.

3.1 Large number of components

From now on X will denote the variety $\mathbb{P}^2_\kappa \setminus D$ for an algebraic closed field κ of characteristic zero and a reduced normal crossing divisor $D \in \mathrm{Div}(\mathbb{P}^2)$. From the description given in the preceding part X is of *log-general type* if and only if

$\deg D \geq 4$. Hence the split case of Lang-Vojta Conjecture for function fields (cfr. Conjecture 10) reads as follows:

Conjecture 11. *Let D be a reduced projective plane curve with normal crossing singularities and let $\tilde{\mathcal{C}}$ be a smooth projective curve. Let S be a finite set of points on $\tilde{\mathcal{C}}$ and $\mathcal{C} = \tilde{\mathcal{C}} \setminus S$ the corresponding affine curve. If D has degree greater than 3 then there exists a bound for the images of non-constant morphisms $\phi : \mathcal{C} \to \mathbb{P}^2 \setminus D$ in terms of the Euler characteristic of $\mathcal{C}$.*

The normal crossing hypothesis here, as in all the versions of Lang-Vojta Conjecture stated before, is essential: we will see in section 3.2 an explicit example due to Corvaja and Zannier of a divisor of degree four without normal crossing singularities where the conclusion of Conjecture 11 does not hold.

It turned out that the problem becomes more difficult the less the irreducible components of the divisor D are; as an example, no case of Conjecture 11 is known for a specific irreducible quartic (see the last part of the Thesis for results into this direction). In this section we are going to review the proof of the Lang-Vojta Conjecture for complements of a completely reducible quartic. This will serve as an example for Corvaja and Zannier generalization to the three component case which will be the basis of our results presented in the next chapter.

Notations. We introduced a useful notation which we are going to use throughout this and the following chapter. For a smooth projective curve $\tilde{\mathcal{C}}$ and a finite set of points S, in analogy with the number field case, we call *S-integers* the elements of the ring of regular function in the affine curve $\mathcal{C} = \tilde{\mathcal{C}} \setminus S$, i.e. elements of $\kappa[\mathcal{C}]$. This functions will have all their poles contained in set S and their ring will be denote by $\mathcal{O}_S$. In the same spirit, invertible elements of the ring of S-integers will be called *S-units*: these are functions with zeros and poles contained in the set S. The group of S-units will be denoted by $\mathcal{O}_S^*$.

3.1.1 The four line case

In this subsection we are going to deal with the case in which D has degree four and four irreducible components, i.e. D consists of four lines in general position. We have already seen a proof for number fields of degeneracy of S-integral points, see Corollary 1.3.4 at page 25. The same argument for function fields, mutatis mutandis, would give the corresponding result. However we will see how this results follows from an extension of Mason's ABC Theorem for polynomials obtained by Brownawell and Masser [BM] and, independently by Voloch [Vol]. The theorem we will prove is the following:

Theorem 3.1.1. *Let D be the divisor of $\mathbb{P}^2$ consisting of four lines in general position. Then for every smooth projective curve $\tilde{\mathcal{C}}$ and a finite set of points $S \subset \tilde{\mathcal{C}}$ there exists a bound for the image of non-constant morphisms $\varphi : \mathcal{C} \to \mathbb{P}^2 \setminus D$ in terms of the Euler characteristic of $\mathcal{C}$, where $\mathcal{C} = \tilde{\mathcal{C}} \setminus S$.*

Proof. Given coordinates $[x_0 : x_1 : x_2]$ for $\mathbb{P}^2$, without loss of generality we can assume that the equations of the four lines are

$$D_1 : x_0 = 0 \qquad\qquad D_2 : x_1 = 0$$
$$D_3 : x_2 = 0 \qquad\qquad D_3 : x_1 + x_2 = x_0.$$

Now morphisms $\mathcal{C} \to \mathbb{P}^2 \setminus D$ corresponds to morphisms

$$\mathcal{C} \to \mathbb{A}^2 \setminus (L_1 + L_2 + L_3),$$

where the affine lines have equations

$$L_1 : x = 0 \qquad L_2 : y = 0 \qquad L_3 : x + y = 1,$$

Here the affine plane $\mathbb{A}^2$ is viewed as $\mathbb{P}^2 \setminus D_1$. Hence the morphism $\varphi : \mathcal{C} \to \mathbb{A}^2 \setminus L$, where $L = L_1 + L_2 + L_3$, is given by $P \mapsto (f(P), g(P))$ for a couple of regular functions on the affine curve $\mathcal{C}$, since φ avoids the line D_1. The fact that it avoids L_1 and L_2 implies that f and g are invertible elements of $\kappa[\mathcal{C}]$, i.e. are S-units. In the same way the function $1-f-g$ is an S-unit as well. Moreover the three S-units satisfy the equation

$$f+g+(1-f-g)=1.$$

Thus the problem is equivalent to studying solutions in S-units to the in-homogeneous equation

$$u_1+u_2+u_3=1$$

and the result follows from the generalized S-unit Theorem of Brownawell-Masser in [BM] (or equivalently using Voloch's result in [Vol]. Moreover, differently from the number field case, for function fields there exists explicit bounds with effective constant for the height that, in this case, provided that no subsum of $f+g+(1-f-g)$ vanishes, reads as follows:

$$h(f,g,1-f-g)\leq 3\max\{0,2g-2+\#S\}.$$

□

From the previous Theorem one can see how the *geometric* property of boundness of the degree for affine curves in the affine surface $\mathbb{P}^2\setminus D$ has been translated into an *arithmetic* property of a set of S-units. In particular methods dealing with solutions of S-unit equations can be applied to recover results related to the geometry of morphisms $\mathcal{C}\to\mathbb{P}^2\setminus D$. This will be a general principle in the subsequent generalization of Theorem 3.1.1 to the case of fewer components for the divisor D.

3.2 The split case

Before moving to the case of three components we want to analyze the peculiar attributes of the previous Theorem related to the general Conjecture 11 focusing on the field of definition of the algebraic varieties involved. In Theorem 3.1.1 the divisor D has been chosen in a way such that the corresponding S-unit equation coming from its irreducible components had a specific shape. This has been possible due to the fact that a degree four and four component divisor has no moduli, i.e. there exists a unique isomorphism class. This implies the following corollary of Theorem 3.1.1:

Corollary 3.2.1. *Let $\tilde{\mathcal{C}}, S, \mathcal{C}, D$ as before. For every fibered threefold $\mathcal{X} \to \mathcal{C}$ such that each fiber is isomorphic to $\mathbb{P}^2 \setminus D$ there exists a bound for the degree of sections $\mathcal{C} \to \mathcal{X}$ in terms of $\chi_S(\mathcal{C})$.*

Proof. By the above discussion each fiber is isomorphic and hence the fibration $\mathcal{X} \to \mathcal{C}$ is isotrivial (independently of the base $\mathcal{C}$). In particular by Lemma 2.1.9 at page 35 there exists a (finite) cover $\mathcal{D} \to \mathcal{C}$ such that the base changed fibration $\mathcal{X}_{\mathcal{D}} := \mathcal{X} \times_{\mathcal{C}} \mathcal{D}$ is trivial. Then each section of the trivial fibration give rise to a map $\mathcal{D} \to \mathbb{P}^2 \setminus D$ for which a bound of the degree is given by Theorem 3.1.1. The conclusion follows from the observation that height of $k(\mathcal{D})$-points is proportional to the height of corresponding $k(\mathcal{C})$-points via the degree of the extension $[k(\mathcal{D}) : k(\mathcal{C})]$ and hence the corresponding degree for images of sections satisfies a similar relationship. $\square$

The difference between the settings of Corollary 3.2.1 and Theorem 3.1.1 is that for the former, the divisor D is assumed to be defined over the function field $k(\mathcal{C})$ whereas for the latter D is defined over the base field $\mathbb{C}$. Being D a divisor without moduli, at most after a cover $\mathcal{D} \to \mathcal{C}$ the two situations coincide. This will not be true in general for log-general type surfaces defined over function fields. However we can explicitly state a characterization of the so called *split case*.

Definition 3.2.2. *Given a fibered threefold $\mathcal{X} \to \mathcal{C}$ where $\mathcal{C} = \tilde{\mathcal{C}} \setminus S$ and the fiber $\mathcal{X}_P \simeq \tilde{\mathcal{X}} \setminus D$ as before, with $\tilde{\mathcal{X}}$ a non-singular projective surface and D a divisor with normal crossing singularities, we say that $\mathcal{X}$ is* split *if the field of definition of D is $\mathbb{C}$. This corresponds to fibration $\mathcal{X} \to \mathcal{C}$ where the fibers are* *all isomorphic, or equivalently, there exists a base change given by a covering $\mathcal{D} \to \mathcal{C}$ such that the fibration is trivial, i.e. that the family is isotrivial. Similarly the* non-split *situation corresponds to a non isotrivial fibration or equivalently to the fact the field of definition of D is not the base field.*

As Corollary 3.2.1 showed, if the divisor D has no moduli, i.e. there exists only one isomorphism class for D, the split case and the non-split case coincide. That will not be the case in general; in particular Conjecture 11 addresses the split case while a non-split case should deal with sections of non isotrivial fibration over a curve.

3.2.1 The conic and the two lines

Consider now the problem in which D is a union of a conic D_1 and two lines D_2, D_3: the normal crossing condition on D is equivalent to the fact that both lines are not tangent to D_1 and the points of intersection $D_1 \cap D_2$, $D_1 \cap D_3$ and $D_2 \cap D_3$ are distinct. For such divisors Conjecture 11 predicts a bound for images of maps $\mathcal{C} \to X = \mathbb{P}^2 \setminus D$ in terms of $\chi_S(\mathcal{C})$. Notice that in this case D has moduli and hence the split case deals with a different situation than the non-split one.

In [CZ5] Corvaja and Zannier solved this case of Conjecture 11 proving the following

Theorem 3.2.3 (Corvaja, Zannier)**.** *Let $\tilde{\mathcal{C}}$ be a smooth complete projective curve, $S \subset \tilde{\mathcal{C}}$ a finite set of points and $D \subset \mathbb{P}^2$ a degree four divisor consisting of a conic and two lines in general position. Let $f : \tilde{\mathcal{C}} \to \mathbb{P}^2$ be a non-constant morphism such that $f^{-1}(D) \subset S$. Then the degree of the curve $f(\tilde{\mathcal{C}})$ verifies*

$$\deg(f(\tilde{\mathcal{C}})) \leq 2^{15} \cdot 35 \cdot \max\{1, \chi(\tilde{\mathcal{C}} \setminus S)\}. \qquad (3.1)$$

Remark 3.2.4. 1. Morphisms $f : \tilde{\mathcal{C}} \to \mathbb{P}^2$ such that $f^{-1}(D)$ is contained in S corresponds bijectively to morphisms $\mathcal{C} \to \mathbb{P}^2 \setminus D$. In particular the Theorem implies Conjecture 11 for such divisors D.

2. As mentioned at the beginning of this Chapter the normal crossing hypothesis is essential. In particular the same authors proved that in the complement of a conic and two lines, where the lines intersect in a point of the conic, there exists curves with vanishing Euler Characteristic and arbitrary large degree (this is Proposition 4.3 of [CZ5]). The proof is effective showing an explicit example of curves $\mathbb{G}_m \to \mathbb{P}^2 \setminus D$ of degree n for every n.

Here is an outline of the strategy used in the proof:

Sketch of the proof. Giving a map $f : \tilde{\mathcal{C}} \to \mathbb{P}^2$ such that $f^{-1}(D)$ is contained in S is equivalent to giving a map $f : \mathcal{C} \to \mathbb{A}^2$ where $\mathbb{A}^2$ is viewed as $\mathbb{P}^2 \setminus D_3$. Such a map is of the form $P \mapsto (x(P), y(P))$ for a choice of coordinates on $\mathbb{A}^2$ such that the affine equations for D_1 and D_2 are

$$D_1 : y^2 = x^2 + \lambda x + 1 \qquad\qquad D_2 : x = 0,$$

for a constant λ (hence D_3 is the line at infinity with respect to this choice of coordinates). In the same way done for 3.1.1 the rational function x, y give rise to the following S-units:

$$u_1 := x \qquad\qquad u_2 := y^2 - x^2 - \lambda x - 1.$$

Similarly $y \in \mathcal{O}_S$ being regular on the affine curve $\mathcal{C}$. Moreover the two S-units and the S-integer verify the following equation

$$y^2 = u_1^2 + \lambda u_1 + u_2 + 1.$$

The proof then proceeds by proving that each solution to the previous equation has bounded height which gives the desired result by the fact that, for the map $f(P) = (x(P), y(P))$, one has

$$\deg f(\mathcal{C}) \leq H_{\tilde{\mathcal{C}}}(x) + H_{\tilde{\mathcal{C}}}(y).$$

Height boundness is proven by differentiate the equation by a specific differential form without multiple poles and observing that the derived equation has many zeros in common with the original one. The problem can be translated in a GCD problem for $(u-1, v-1)$ for S-units u, v and then a Theorem by the same authors gives the full conclusion. $\square$

We state here the detailed version of the lemma on the existence of a "proper derivation" on the curve C because we are going to need it in the next chapter.

Lemma 3.2.5 (Corvaja,Zannier [CZ5]). *For every smooth projective curve $\tilde{C}$ of genus $g(\tilde{C})$ and every finite set of points $S \subset \tilde{C}$, there exists a differential form $\omega \in \tilde{C}$ and a finite set $T \subset \tilde{C}$ of cardinality $\sharp T = \max\{0, 2g(\tilde{C}) - 2\}$ such that for every $u \in \mathcal{O}_S^*$ there exists an $(S \cup T)$-integer $\theta_u \in \mathcal{O}_{S \cup T}$ having only simple poles such that*

$$\frac{d(u)}{u} = \theta_u \cdot \omega \qquad\qquad H_{\tilde{C}}(\theta_u) \leq \chi_S(\tilde{C}). \tag{3.2}$$

Moreover if $a \in \mathcal{O}_S$ then there exists an $a' \in \mathcal{O}_{S \cup T}$ such that

$$d(a) = a' \cdot \omega \qquad\qquad H_{\tilde{C}}(a') \leq H_{\tilde{C}}(a) + \chi_S(\tilde{C}).$$

Sketch of the proof. If $g = g(\tilde{C}) > 0$ then there exists a regular $\tilde{\omega} \in \Omega = \Omega^1(\tilde{C})$ and hence one can define a 1-form ω with $2g - 2$ distinct simple zeros. Now for each S-unit u, the rational function $\theta = \theta_u$ satisfying 3.2 has poles either in the zeros of ω or in S. Defining T as the set of zeros of ω makes θ a $S \cup T$-unit which obviously verifies

$$H_{\tilde{C}}(\theta_u) \leq \chi_S(\tilde{C}).$$

If $g = 0$, given the fact that $\sharp S \geq 2$, there exists a differential form ω without zeros and with two distinct poles at two S-points. Now by the same argument as before the set of poles pf θ is at most $\sharp S - 2$ as wanted. The result for S-integers follows the same ideas. $\square$

Example 3.2.6. As an example of application of the previous Lemma consider the simpler equation

$$y^2 = u_1 + u_2 + 1,$$

with, as before, $u_1, u_2 \in \mathcal{O}_S^*$ and $y \in \mathcal{O}_S$. Deriving with respect to the differential form given by Lemma 3.2.5 gives

$$2yy' = u_1' + u_2' = \frac{u_1'}{u_1}u_1 + \frac{u_2'}{u_2}u_2.$$

With some manipulation one gets the following two equalities:

$$\begin{aligned}\frac{u_1'}{u_1}y^2 - 2yy' &= -\frac{u_1'}{u_1}(w_2 - 1)\\ \frac{u_2'}{u_2}y^2 - 2yy' &= -\frac{u_2'}{u_2}(w_1 - 1),\end{aligned}$$

where

$$\begin{aligned}w_1 &:= u_1\left(\frac{u_1'}{u_1} - \frac{u_1}{u_1'} - 1\right)\\ w_2 &:= u_2\left(\frac{u_2'}{u_2} - \frac{u_1'}{u_1}\right)\frac{u_1}{u_1'} - 1.\end{aligned}$$

Hence by the fact that y divides both $-\frac{u_1'}{u_1}(w_2 - 1)$ and $-\frac{u_2'}{u_2}(w_2 - 1)$ we are led to consider the greatest common divisor between $(w_1 - 1)$ and $(w_2 - 1)$. Lemma 3.2.5 implies that the height of the functions w_1 and w_2 are bounded in terms of S, T and g but not in terms of u_1 and u_2, a feature which will be critical in the subsequent applications.

3.2.2 Greatest Common Divisor estimates

From the sketch of the proof for Theorem 3.2.3 and the last example, one can see how algebraic hyperbolicity for $\mathbb{P}^2 \setminus D$ can be translated intro a problem related to divisibility inside the ring of S-units. In this direction the main theorem by Corvaja and Zannier reads as follows

Theorem 3.2.7 (Corvaja, Zannier [CZ5]). *Let $a, b \in \mathcal{O}_S^*$ be multiplicatively independent non-constant S-units, and let α, β be* *positive integers. Then one of the following condition holds:*

- $H(a) \leq \alpha[k(\tilde{\mathcal{C}} : k(a,b)]$ *and* $H(b) \leq \beta[k(\tilde{\mathcal{C}} : k(a,b)]$

- $$\sum_{v \notin S} \min\{v(1-a), v(1-b)\} \leq \frac{(\alpha+2\beta)H(b)+\beta H(a)}{\alpha\beta+\alpha+\beta} + \frac{\alpha\beta+\alpha+\beta-1}{2}\chi_S(\tilde{\mathcal{C}}).$$

Remark 3.2.8. Theorem 3.2.7 can be seen as a result on the greatest common divisor between $1-a$ and $1-b$, or better a S-gcd, namely the number of common zeros of the two rational functions outside the set S. In particular an explicit choice for the constants α, β can give an upper bound which depends only on the height of the S-units as well as on the Euler Characteristic of the underline curve $\tilde{\mathcal{C}}$.

Theorem 3.2.9 (Corvaja and Zannier). *Let $a, b \in \mathcal{O}_S^*$ not both constant, and let $H := \max\{H(a), H(b)\}$. Then*

(i) *If a, b are multiplicatively independent, we have*

$$\sum_{v \notin S} \min\{v(1-a), v(1-b)\} \leq 3\sqrt[3]{2}(H(a)H(b)\chi(\mathcal{C}))^{\frac{1}{3}} \leq 3\sqrt[3]{2}(H^2\chi(\mathcal{C}))^{\frac{1}{3}}. \tag{3.3}$$

(ii) *If a, b are multiplicatively dependent, let $a^r = \mu b^s$ be a generating relation. Then either $\mu \neq 1$ and*

$$\sum_{v \notin S} \min\{v(1-a), v(1-b)\} = 0,$$

or $\mu = 1$ and

$$\sum_{v \notin S} \min\{v(1-a), v(1-b)\} \leq \min\left\{\frac{H(a)}{|s|}, \frac{H(b)}{|r|}\right\} \leq \frac{H}{\max\{|r|, |s|\}}. \tag{3.4}$$

Recall that a *generating relation* for two multiplicative dependent elements is a relation $a^r = \lambda b^s$ such that the pair (r, s) generates the lattice of all the pairs (ρ, σ) such that $a^\rho = \lambda b^\sigma$. This result is the function field analogue of a theorem by the same authors obtained in the arithmetic case (see [CZ5]).

We end this section mentioning a link between these results and Lang-Vojta Conjecture: as pointed out by Silverman in [Sil] this link is profound and goes beyond the applications to function fields arithmetic. For this let us briefly recall the definition of *generalized (logarithmic) greatest common divisor* in a number field:

Definition 3.2.10 (Silverman [Sil], Corvaja and Zannier [CZ3]). *Given a, b in a number field k, the* greatest common divisor *of a, b is defined to be*

$$h_{\mathrm{gcd}}(a, b) = \sum \min\{\max\{0, v(a)\}, \max\{0, v(b)\}\},$$

where the sum runs over all the places of k. An S-gcd will be defined in the same way by summing over all the places outside the finite set S.

Silverman noticed that the former is actually a Weil Height in the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ over $(0, 0)$ with respect to the exceptional divisor. From this observation one can prove that Vojta Conjecture 5 for such a blow-up (with respect to a suitable ample divisor and with some natural hypotheses) implies an upper bound for the greatest common divisor of a couple of algebraic points (this is Theorem 6 of [Sil]). Therefore, in some sense, a gcd argument applied to Lang-Vojta Conjecture is very natural.

3.3 Extension to ramified covers of $\mathbb{G}_m^2$

The fundamental characteristic of divisors D in Theorem 3.2.3 was the presence of *three* irreducible components: in particular the three components implies the existence of S-units and an S-integer satisfying an equation whose solutions (and their heights) lie at the core of Corvaja and Zannier methods. Thus it seems natural to ask if similar results could be obtained for complements of three component divisors in more general surfaces other than $\mathbb{P}^2$. It turns out that is true and is the content of a recent generalization of Theorem 3.2.3 by the same authors obtained in [CZ7] which we are going to recall because of its implications with the results of the next chapter.

The starting point is the following easy observation:

Remark 3.3.1. Let $D = D_1 + D_2 + D_3$ be the divisor in $\mathbb{P}^2$ formed by a conic and two lines in general position. Let $f_i = 0$ be the equation of the i-th irreducible component assuming $\deg D_1 = 2$. Then the following map $\mathbb{P}^2 \setminus D \to \mathbb{G}_m^2$ to the bi-dimensional torus is dominant:

$$P \mapsto \Big(\frac{f_1}{f_3^2}(P), \frac{f_2}{f_3}(P)\Big).$$

It turns out that the map defined above is the only feature required for algebraic hyperbolicity of some log-general type surface.

Theorem 3.3.2 (Corvaja and Zannier [CZ7]). *Let X be a smooth affine surface with a finite map $\pi : X \to \mathbb{G}_m^2$ such that the closure of the image of the ramification divisor Z does not pass through the singular points of the boundary of $\mathbb{G}_m^2$. If X is of log-general type then there exists a constant $\gamma = \gamma(X, \pi)$ such that each affine curve $C \subset X$ verifies:*

$$\deg C \leq \gamma \cdot \max\{1, \chi_C\}.$$

From the previous Theorem and the remark follows a result for complements of $\mathbb{P}^2$ that generalizes Theorem 3.2.3 encompassing more general cases. The result reads as follows

Theorem 3.3.3 (Corvaja and Zannier). *Let D a plane curve with normal crossing singularities of degree at least four and at* *least three irreducible components. Then there exists a constant $\delta = \delta(D)$ such that each affine curve $\mathcal{C}$ in $\mathbb{P}^2 \setminus D$ verifies*

$$\deg \mathcal{C} \leq \delta \max\{1, \chi_{\mathcal{C}}\}.$$

Clearly this result implies Theorem 3.2.3 and Theorem 3.1.1 although with possibly different constants involved.

3.3.1 Extra divisor from ramification

The proof of Theorem 3.3.2 and hence of Theorem 3.3.3 relies on an estimate of the contribution of the ramification divisor to the degree of the affine curves in the surfaces. The ideas behind the proof are the following:

1. The starting point is that for each map $\mathcal{C} \to X$ the contribution of the ramification divisor to its degree is small. More precisely let $\varphi : \mathcal{C} \to X$ be a non-constant map from an affine curve $\mathcal{C}$ to the affine surface X, such that its image is not contained in the ramification divisor $Z \subset X$ of the finite map $\pi : X \to \mathbb{G}_m^2$. Then for every $\epsilon > 0$ there exists a constant Γ (which depends on X, π and the Euler characteristic of $\mathcal{C}$) such that for every such φ either the height $H = H(\varphi)$ of φ verifies $H \leq C$ or the pullback divisor $\varphi^*(Z)$ is such that

$$\deg \varphi^*(Z) \leq \epsilon H.$$

   The bound follows from a height bound for zeros of polynomials without repeated factors obtained in [CZ5].

2. From the fact that the affine surface X is of log-general type one can prove that the ramification divisor is big, being linearly equivalent to $K_{\tilde{X}} + \pi^*(E)$ where E is the divisor $\mathbb{P}^2 \setminus \mathbb{G}_m^2$, whose bigness is equivalent to X be a surface of log-general type. Here π denotes also the "completed" map $\pi : \tilde{X} \to \mathbb{P}^2$ and E is viewed as a divisor on $\mathbb{P}^2$.

Being big, Z is in the closure of the effective cone of X and hence it's intersection with every curve $\tilde{C}$ in X is bounded below by a constant times the degree of $\tilde{C}$ (a part from a finite set of curves).

3. Now the intersection of the image $\varphi(\tilde{C})$ with the (completion of) the ramification divisor $\tilde{Z}$ is bounded from below by the previous argument and at the same time its affine part is bounded from above. This implies that the most part of the intersection should be concentrated on the points at infinity, i.e. on S. This can be reformulated in terms of height bounds for S-units solutions to a sum of monomials with non zero constant term and therefore follows under the cases of Brownawell-Masser generalized S-units Theorem. In particular this implies that either the S-units, and hence $\varphi(C)$, have bounded height or they verify a dependence relation which gives, again, a bound for the degree of the immersed curve.

The previous sketch of the proof shows that the result has been reduced to an application to Brownawell-Masser Theorem as in the four line case: in particular the ramification divisor Z plays the role of the missing fourth component of D that appears in Theorem 3.1.1. It is expected that similar results holds in the non-split situation using ideas similar to the one exposed in the next chapter.

4

The non-split case

4.1 Statement of the problem

The main goal of this chapter is to generalize the situation of chapter 3 to the so-called non-split case, i.e. the case of Lang-Vojta Conjecture for the complement of a conic and two lines in $\mathbb{P}^2$, where now the divisor is defined over the function field of a curve rather than on $\mathbb{C}$. As in the constant case we are going to reduce the problem to solve an equation and bound the height of its solutions with Corvaja and Zannier method. In our case the equation that describes this setting reads as follows:

$$y^2 = u_1^2 + \lambda(P)u_1 + u_2 + 1. \tag{4.1}$$

Here again y is a S-integer and u_1, u_2 are S-units. We note that this equation is precisely the same considered in [CZ5] where the polynomial in the right-hand side has now non constant coefficients. Geometrically this corresponds to the data of an (affine) threefold X fibered over the curve $\mathcal{C}$ where each fiber is isomorphic to $\mathbb{P}^2 \setminus D$ and D is a divisor consisting of a conic and two lines. Each solution of the equation (4.1) gives a section of the fibration $X \to \mathcal{C}$.

The situation considered in this chapter is made explicit in the following diagram:

$$\begin{tikzcd} X \arrow[d, "\pi"] \\ \mathcal{C} \arrow[u, bend left, "\sigma"] \arrow[r, "\lambda"] & \mathbb{P}^1 \end{tikzcd} \tag{4.2}$$

Here the parameter $\lambda(P)$ is a rational function of the cross-ratio of the four singular points on the conic of the divisor on the fiber over P and σ is a section of the projection π (see 4.3 for a detailed description of the geometric setting). We observe that this is the natural generalization of the settings considered in chapter 3: morphisms from the affine curve $\mathcal{C} = \tilde{\mathcal{C}} \setminus S$ to $\mathbb{P}^2 \setminus D$ can be seen as sections of the trivial $(\mathbb{P}^2 \setminus D)$-bundle over the curve $\mathcal{C}$. In the case considered in this chapter the trivial bundle is replaced by a fibration in which the divisor at infinity is moving. Moreover, generalizing the constant case, the three irreducible components of the divisor $D = D_P$ are not supposed to be in general position for every $P \in \tilde{\mathcal{C}}$ (although we need some restrictions on the "degeneracy" of the divisor).

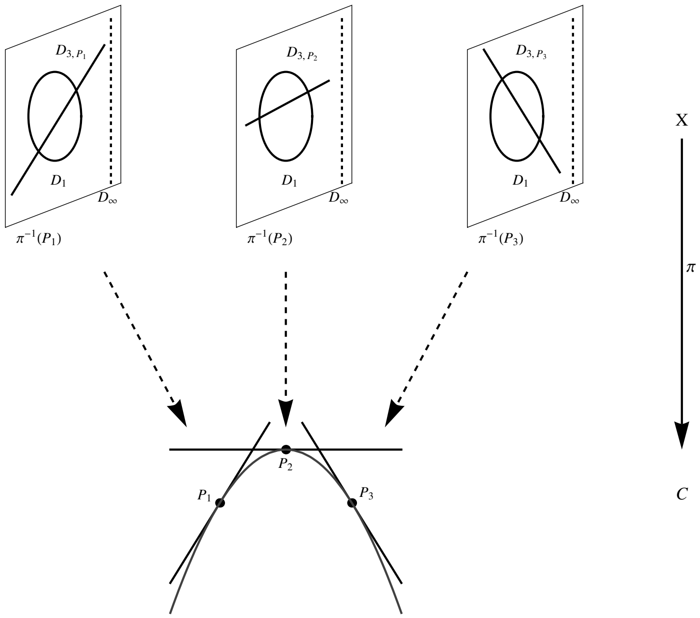
The main result of this chapter is the following

Theorem 4.1.1. *Let $\tilde{\mathcal{C}}, S, X$ as above. Let $\sigma : \mathcal{C} \to X$ be a non constant section for the fibration $\pi : X \to \mathcal{C}$ where each fiber is isomorphic to $\mathbb{P}^2 \setminus D$. Then, in a suitable projective embedding of the variety X, if the fibration is not birationally trivial the degree of the curve $\sigma(\tilde{\mathcal{C}})$ verifies*

$$deg(\sigma(\tilde{\mathcal{C}})) \leq 2^{13} \cdot \big(58 \cdot \chi_S(\tilde{\mathcal{C}}) + 28 H_{\tilde{\mathcal{C}}}(\lambda)\big) + 8 H_{\tilde{\mathcal{C}}}(\lambda).$$

Example 4.1.2. Consider a plane smooth curve $\mathcal{C}$ in $\mathbb{P}^2_{\mathbb{C}}$. For each point $P \in \mathcal{C}$ let t_P denote the tangent line to $\mathcal{C}$ at P. This defines a fibration over $\mathcal{C}$ in the following way: over a point $P \in \mathcal{C}$ let $\mathcal{X}_P$ be the complement in $\mathbb{P}^2$ of the divisor formed by a fixed quadric D_1, the line at infinity D_∞ (assuming a choice of coordinate has been made) and the line $D_{3,P} = t_P$. A picture of this situation can be seen in Figure 4.1.

Figure 4.1: Fibered threefold



The threefold

$$\mathcal{X} = \bigcup_{P \in \mathcal{C}} \mathcal{X}_P \to \mathcal{C}$$

can be seen as a surface defined over the function field of the completion of the normalization $\tilde{\mathcal{C}}$ of $\mathcal{C}$ where a point of the surface $P \in \mathcal{X}(k(\tilde{\mathcal{C}}))$ corresponds to a section $\sigma_P : \tilde{\mathcal{C}} \to \tilde{\mathcal{X}}$ such that $\sigma_P^{-1}(D) \subset \tilde{\mathcal{C}} \setminus \mathcal{C}$. In particular in the case in which the divisor

$$D = \bigcup_{P \in \mathcal{C}} D_1 + D_\infty + D_{3,P}$$

has normal crossing, being $\deg D = 4$, each fiber is of log-general type and hence Theorem 4.1.1 can be applied giving a bound for the degree of images $\sigma_P(\tilde{\mathcal{C}})$ as expected by Conjecture 10.

4.2 Moduli of three components divisors

In this section we will analyze configurations of a conic and two lines in $\mathbb{P}^2$. Our aim is to prove that a moduli space for equivalence classes of these divisors is of dimension one.

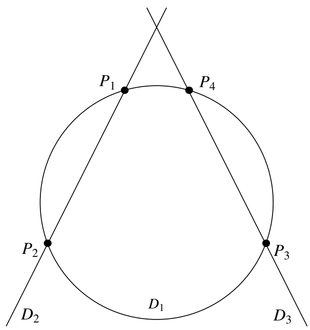


Figure 4.2: Configuration of a conic and two lines in general position

For this let D be the sum of a conic D_1 and D_2, D_3 distinct lines in $\mathbb{P}^2$ defined over κ. This divisor has four singular points, the four points of intersection between D_1 and $D_2 + D_3$; these points are distinct in the case when D_1, D_2, D_3 are in general position, i.e. D has normal crossing singularities. We want to characterize completely isomorphism classes of such divisors.

First we observe that, applying birational automorphisms of $\mathbb{P}^2$ each class possesses a representative with a fixed conic $\overline{D_1}$ as component of degree two. Hence the problem can be reduced to study isomorphism classes of couple of lines not tangent to $\overline{D_1}$ whose intersection is not on the conic. One of such divisor is visible in figure 4.2. Secondly one can notice that the problem is equivalent to the study of fourples of points on $\mathbb{P}^1$, via the isomorphism between the conic and $\mathbb{P}^1$, that give rise to isomorphic divisors (here we take the line D_2 as the one passing through the first two points, and the line D_3 passing through the last two points). In other words a moduli space for our problem will be represented by a scheme with a map from $\mathcal{M}_{0,4} \cong \mathbb{G}_m \setminus \{1\}$, where the last isomorphism is given by the cross-ratio. However, although the cross-ratio of the four points gives information about the divisor it does not characterize completely an isomorphism class. As an example consider the following two fourples (P_1, P_2, P_3, P_4) (P_2, P_1, P_3, P_4): clearly they define the same divisor but the two cross-ratios are inverse of each other. Hence configurations of four points with the same cross-ratio give rise to isomorphic divisors, but the converse is not true as shown by the preceding example. However the following basic lemma holds:

Lemma 4.2.1. *Given two fourples $\underline{P} = (P_1, \ldots, P_4)$ and $\underline{Q} = (Q_1, \ldots, Q_4)$ of points in $\mathbb{P}^1$. At most after applying the map mapping the first two points of each four-ple to $0, \infty$ and applying the basic theory of cross-ratio, we can assume $P_3 = Q_3$ and $P_4 = Q_4$. If neither the two fourples have the same cross-ratio nor $\underline{Q}$ is obtained by $\underline{P}$ by permuting the points, then the configuration of divisors defined by $\underline{P}$ and $\underline{Q}$ are not isomorphic.*

Hence we are reduced to calculate which permutations of four points give rise to isomorphic configurations of divisors. We can then consider the action of the permutations' group S^4 on an ordered set of four points in the projective line, i.e. an element of $(\mathbb{P}^1)^4$; an easy case by case analysis shows that the subgroup of S^4 that leaves the divisor unchanged is $G = \langle (12), (13)(24), (14)(23) \rangle$. Hence, by classic properties of the cross-ratio, the only generator of G that changes the cross-ratio and under which the divisor configuration is invariant is (12). Thus in order to completely describe isomorphism classes of degree four and three components divisors in $\mathbb{P}^2$ it is sufficient to define a map

$$\lambda' : \left\{ \begin{array}{c} \text{degree four and three} \\ \text{components divisor in } \mathbb{P}^2 \end{array} \right\} \longrightarrow \mathbb{P}^1$$

constant on isomorphic divisors. By the description given above we obtain a natural 2:1 map, from the moduli space $\mathcal{M}_{0,4}$ to the moduli space of degree four and three components divisors: this map simply associates to each fourple the two lines passing to the four points. With abuse of notation we indicate as λ' the composition

$$\lambda' : \mathcal{M}_{0,4} \to \left\{ \begin{matrix} \text{degree four and three} \\ \text{components divisor in } \mathbb{P}^2 \end{matrix} \right\} \to \mathbb{P}^1,$$

which is defined as

$$\begin{aligned} \lambda'(P_1, P_2, P_3, P_4) &= \frac{\beta\,(P_1, P_2, P_3, P_4)^2 + 1}{\beta\,(P_1, P_2, P_3, P_4)} - 2 \\ &= \frac{\left(\beta(P_1, P_2, P_3, P_4) - 1\right)^2}{\beta(P_1, P_2, P_3, P_4)}, \end{aligned} \tag{4.3}$$

where β is the cross-ratio. From this definition λ' is a morphism from the (ordered) quadruple of points in the conic D_1 which associates to every configuration of the divisor D a point of $\mathbb{P}^1$. However λ' is a function of the cross-ratio of the quadruple $P_1, \ldots, P_4$ and so it is defined only when there are at least three distinct points. In our situation, requiring that over the affine curve $\mathcal{C}$ the fiber is $\mathbb{P}^2 \setminus D$ and D has four singular distinct points is equivalent to require that the set S contains all the poles and zeros of β and hence all the poles of λ': this implies that some cases of non general position are allowed but only over points in S. We will moreover enlarge S such that it contains all the zeros of λ': this assumptions is made in order to include the case of cross-ratio 2 (given by the factor -2 in the preceding formula) where the quartic has a non-normal crossing singularity, and also to apply Theorem 4.4.6 and has the advantage that there will be no need of distinguish between the case of negative and positive Euler characteristic. At the same time, this is not a strong restriction because λ' will be a datum of the variety we want to deal with and hence it does not depend on the method used for the proof.

With abuse of notation we will sometimes indicate the value $\lambda'(P_1, \dots, P_4)$ as $\lambda'(D)$ where the configuration of D is defined by the points $P_1, \dots, P_4$ on the conic D_1.

4.3 Affine threefolds

We are interested in a specific class of affine threefolds fibered over affine curves which generalizes the trivial $\mathbb{P}^2 \setminus D$-bundle considered in the split case. More in detail we consider the following class of affine threefolds:

> $(\star)$ X is an affine threefold fibered over the affine curve $\mathcal{C}$ such that the completion of the fibration is the trivial $\mathbb{P}^2$-bundle over $\tilde{\mathcal{C}}$. Every fiber $\pi^{-1}(P)$ for a point $P \in \tilde{\mathcal{C}}$ is of the form $\mathbb{P}^2 \setminus D_P$ where D_P is a divisor of $\mathbb{P}^2$ of degree four formed by an irreducible conic and two lines such that there are at least three distinct singular points. If the point P is in $\mathcal{C}$ then the function λ' is regular on D_P.

(see the diagram 4.2) . As an example of this situation one can consider the bundle $\pi : X \to \mathcal{C}$ where $\tilde{\mathcal{C}}$ is a plane rational curve and the divisor D_P is formed by the conic $x^2 + y^2 = 1$, the line at infinity (in affine coordinates) and the line t_P, i.e. the tangent line to the curve at the point P. In this example S will contain every point $P \in \tilde{\mathcal{C}}$ such that D_P is a pole for λ'.

It follows from the definition of the class $(\star)$ that giving such a threefold is equivalent to giving a rational function

$$\lambda : \tilde{\mathcal{C}} \dashrightarrow \mathbb{P}^1,$$

which associates to a point $P \in \mathcal{C}$ a point of $\mathbb{P}^1$ viewed as the value of the function $\lambda'(D_P)$, i.e. $\lambda(P)$ specifies the isomorphism class of the divisor D_P in the fiber over P. More in detail the affine threefold will be determined by the properties of the divisor $D = \cup D_P$ which can be described as follows: we can naturally embed X inside $\tilde{X} := \tilde{\mathcal{C}} \times \mathbb{P}^2$ and denote by $p_1 : \tilde{X} \to \tilde{\mathcal{C}}$ and $p_2 : \tilde{X} \to \mathbb{P}^2$ the two projections. Then the fibration $X \to \mathcal{C}$ is uniquely determined by a line bundle $\mu \in \mathrm{Pic}(\tilde{\mathcal{C}})$ and the choice of a divisor $D \in |p_1^*(\mu) \otimes p_2^*(\mathcal{O}_{\mathbb{P}^2}(4))|$, see diagram below.

$$\begin{tikzcd}
 & p_1^*(\mu) \otimes p_2^*(\mathcal{O}_{\mathbb{P}^2}(4)) \arrow[d] & \\
X \arrow[r, hook] \arrow[d, "\pi"] & \tilde{\mathcal{C}} \times \mathbb{P}^2 \arrow[r, "p_2"] \arrow[d, "p_1"] & \mathbb{P}^2 \\
\mathcal{C} \arrow[r, hook] \arrow[u, bend left, "\sigma"] & \tilde{\mathcal{C}} &
\end{tikzcd} \tag{4.4}$$

However not every divisor in the linear system gives rise to a fibration satisfying condition $(\star)$: we impose the condition that $D|_{\pi^{-1}(P)}$ has three components. This in particular implies that on every fiber the divisor is determined by the value of a function of the map λ' defined above, i.e. over every point $P \in \mathcal{C}$ the fiber is uniquely specified by the value of λ' on the singular points of D_P, which we assume to be at least three, and such that the cross-ratio has no pole for this configuration.

In particular we will prove that every threefold satisfying $(\star)$, described by a non constant rational map $\lambda : \tilde{\mathcal{C}} \dashrightarrow \mathbb{P}^1$, has images of sections with bounded degree in terms of the Euler Characteristic of the base curve.

4.4 Proof of the result

From now on we will work on an affine algebraic variety of dimension three satisfying condition $(\star)$. We will denote by D_P the divisor defined on the fiber over the point P (or sometimes just D where the point we refer to is clear) and its three irreducible components will be indicated by D_1 (the conic) and D_2, D_3 (the two lines). The function $\lambda : \tilde{\mathcal{C}} \to \mathbb{P}^1$ will denote the map defined by

$$P \mapsto \lambda'(D_P).$$

We will suppose, at most after enlarging S, that λ is a S-unit, i.e. S contains all its zeros and poles. We begin by proving the following:

Lemma 4.4.1. *Let $\tilde{\mathcal{C}}, S$ be as before and let $\pi : X \to \mathcal{C}$ be an affine fibered threefold verifying condition $(\star)$ and characterized by a non-constant rational function λ. Let $\sigma : \mathcal{C} \to X$ be a section of π. Then there exist S-units $u_1, u_2 \in \mathcal{O}_S^*$ and an S-integer $y \in \mathcal{O}_S$ satisfying*

$$y^2 = u_1^2 + \lambda u_1 + u_2 + 1 \tag{4.5}$$

and such that $\deg \sigma(\mathcal{C}) \leq H_{\tilde{\mathcal{C}}}(u_1) + H_{\tilde{\mathcal{C}}}(y)$.

Proof. From condition $(\star)$ it follows that, after a choice of homogeneous coordinates, we can consider affine coordinates (x, y) in every fiber with respect to the line D_2 viewed as the line at infinity $x_0 = 0$. In this system of coordinates, without loss of generality, the line D_3 has equation $x = 0$ and the conic D_1 has equation $y^2 = x^2 + \lambda x + 1$. Now we turn our attention to the section $\sigma : \tilde{\mathcal{C}} \setminus S \to X$. In our setting σ can be written as

$$\sigma(P) = (x(P), y(P), P) \in \pi^{-1}(P) \cong \mathbb{P}^2 \setminus D_P.$$

Now it is a general fact that such a morphism has degree bounded by the height of its components: indeed the degree of σ is the number of intersection points with a generic hyperplane in a projective space where $\sigma(\tilde{\mathcal{C}})$ is embedded and this number is bounded by the sum of the heights of the components x and y. This proves that

$$\deg \sigma(\mathcal{C}) \leq H_{\tilde{\mathcal{C}}}(u_1) + H_{\tilde{\mathcal{C}}}(y),$$

where $u_1 := x$. The fact that the image $\sigma(P)$ avoids the line D_2 means that the function $u_1 := x \in \mathcal{O}_S^*$, i.e. it is a unit, and $y \in \mathcal{O}_S$, i.e. it is a regular function on the affine curve $\mathcal{C}$. Moreover, the condition that the image of σ avoids also the conic D_1 in every fiber means that we can define another S-unit u_2 where

$$u_2 = y^2 - u_1^2 - \lambda u_1 - 1.$$

Hence the units u_1, u_2 and the S-integer y verify equation (4.5). $\square$

We will now work with equation (4.5) in order to describe its solutions. Our goal is to prove the following

Theorem 4.4.2. *With the notation above, every solution*

$$(y, u_1, u_2) \in \mathcal{O}_S \times (\mathcal{O}_S^*)^2$$

of equation (4.5) satisfies one of the following conditions:

(i) *a sub-sum on the right term of (4.5) vanishes;*

(ii) *u_1, u_2 verify a multiplicative dependence relation of the form $u_1^r \cdot u_2^s = \mu$, where $\mu \in \kappa^*$ is a scalar and $r, s,$ are integers, non both zeros such that* $\max\{r, s\} \leq 5$;

(iii) *the following bound holds:*

$$\max\{H_{\tilde{\mathcal{C}}}(u_1), H_{\tilde{\mathcal{C}}}(u_2)\} \leq 2^{12}\big(58\chi_S(\tilde{\mathcal{C}}) + 28H_{\tilde{\mathcal{C}}}(\lambda)\big) + 8H_{\tilde{\mathcal{C}}}(\lambda).$$

We will now follow the proof, given by Corvaja and Zannier of the constant case (Theorem 3.2.3 at page 50), deepening the differences between the present and the non-split situation. The starting point is the description of a suitable notion of derivatives of rational function over the curve $\mathcal{C}$. This comes from Lemma 3.2.5 (page 52) where an appropriate differential form is defined. We just notice that the Lemma refers to the curve only without any reference to the bundle and hence can be applied in all the cases under consideration.

From now on the differential form ω and the finite set T appearing in Lemma 3.2.5 will be fixed and, for a rational function $a \in \kappa(\tilde{\mathcal{C}})$ we will denote by a' the only rational function such that $d(a) = a' \cdot \omega$.

We consider now the derivative of a polynomial $A \in \kappa[X, Y]$ calculated in a point u_1, u_2 for some S-units u_1, u_2. One can prove that (see [CZ5] Lemma 3.7)

$$(A(u_1, u_2))' = B(u_1, u_2),$$

where

$$B(X, Y) = \frac{u_1'}{u_1} \cdot X \frac{\partial A}{\partial X}(X, Y) + \frac{u_2'}{u_2} \cdot Y \frac{\partial A}{\partial Y}(X, Y).$$

We will use this identity in order to deal with equation (4.5).

Lemma 4.4.3. *Let*

$$\begin{aligned} A(X, Y) &= X^2 + \lambda X + Y + 1, \\ B(X, Y) &= 2\,\frac{u_1'}{u_1}\,X^2 + \lambda\left(\frac{u_1'}{u_1} + \frac{\lambda'}{\lambda}\right) X + \frac{u_2'}{u_2}\,Y \end{aligned} \tag{4.6}$$

be polynomials in $\mathcal{O}_{S\cup T}(\tilde{\mathcal{C}})[X, Y]$, and let $F(X) \in \mathcal{O}_{S\cup T}[X]$, $G(Y) \in \mathcal{O}_{S\cup T}[Y]$ be the resultants of $A(X, Y), B(X, Y)$ with respect to Y and X, i.e. the polynomials

$$F(\mathbf{X}) = \mathbf{X}^2\left(2\frac{u_1'}{u_1} - \frac{u_2'}{u_2}\right) + \mathbf{X}\left(\frac{u_1'}{u_1} - \frac{u_2'}{u_2} + \frac{\lambda'}{\lambda}\right)\lambda - \frac{u_2'}{u_2} \tag{4.7}$$

$$\begin{aligned} G(\mathbf{Y}) = \mathbf{Y}^2\left(2\frac{u_1'}{u_1} - \frac{u_2'}{u_2}\right)^2 + \mathbf{Y}\Bigg[\left(\frac{u_1'}{u_1}\right)^2(8 - \lambda^2) + \frac{u_1'}{u_1}\frac{u_2'}{u_2}(\lambda^2 - 4) + \\ + \lambda\lambda'\left(\frac{\lambda'}{\lambda} - \frac{u_2'}{u_2}\right)\Bigg] + \left(\frac{u_1'}{u_1}\right)^2(4 - \lambda^2) + \lambda'^2. \end{aligned} \tag{4.8}$$

Then for every solution $(y, u_1, u_2) \in \mathcal{O}_S \times (\mathcal{O}_S^*)^2$ of (4.5) we have

$$\begin{aligned} y^2 &= A(u_1, u_2), \\ 2yy' &= B(u_1, u_2). \end{aligned}$$

Moreover the S-integer y divides both $F(u_1)$ and $G(u_2)$ in the ring $\mathcal{O}_{S\cup T}$.

Proof. Obviously equation (4.5) is exactly $y^2 = A(u_1, u_2)$. Moreover $A(u_1, u_2)' = B(u_1, u_2)$, so we have $2yy' = B(u_1, u_2)$ as desired.

For the second fact we observe that, for the general theory of resultants, F and G are linear combinations of A and B with coefficients that are polynomials in $\mathcal{O}_{S\cup T}$, concluding the proof. □

Our next step will be to factor the polynomials $F(X), G(Y)$ in a suitable finite field extension of $\kappa(\tilde{\mathcal{C}})$; this extension will be a function field $\kappa(\tilde{\mathcal{D}})$ for a cover $\tilde{\mathcal{D}} \to \tilde{\mathcal{C}}$. Besides, we will estimate the Euler characteristic of the curve $\tilde{\mathcal{D}}$. From now on we will suppose that the leading and the constant term of the polynomial $F(X), G(Y)$ are both non zero.

Lemma 4.4.4. *Given $F, G, \tilde{\mathcal{C}}, S, T$ as before, there exists a cover $\tilde{\mathcal{D}} \to \tilde{\mathcal{C}}$, of degree less or equal to four, such that the Euler characteristic of $\tilde{\mathcal{D}} \setminus U$ verifies*

$$\chi_U(\tilde{\mathcal{D}}) \leq 53\chi_S(\tilde{\mathcal{C}}) + 28H_{\tilde{\mathcal{C}}}(\lambda) + 5 \cdot max\{0, 2g(\tilde{\mathcal{C}}) - 2\}, \quad (4.9)$$

where U is the set formed by the pre-images of the zeros of the leading and constant coefficients of F and G and the pre-images of S and T.

Proof. Our goal was to factor $F(X)$ and $G(X)$, so we define the cover $p : \tilde{\mathcal{D}} \to \tilde{\mathcal{C}}$ by the property that $\kappa(\tilde{\mathcal{D}})$ is the splitting field of $F(X) \cdot G(X)$ over $\kappa(\tilde{\mathcal{C}})$. From this definition it is straightforward that $\deg p$ is at most four, because $\kappa(\tilde{\mathcal{D}})$ is generated over $p^*(\kappa(\tilde{\mathcal{C}}))$ by the square roots of the discriminants of the two polynomials (recall that $F(X)$ and $G(X)$ both have degree 2).

We will now bound the Euler characteristic of $\tilde{\mathcal{D}} \setminus U$ via the Riemann-Hurwitz formula; for this goal we need an estimate of the ramification points of the cover p. First of all we notice that the ramification points are all contained in the zeros and poles of the discriminants; moreover at any point the ramification index is at most two. The poles are contained in $S \cup T$ and the number of zeros of the discriminants is bounded by their heights. The discriminant of $F(X)$ is

$$\begin{aligned} Discr(F(X)) = \left(\frac{u_2'}{u_2}\right)^2 (\lambda^2 - 4) + \left(\frac{u_2'}{u_2}\right)\left(8\frac{u_1'}{u_1} - 2\frac{u_1'}{u_1}\lambda^2 + \right. \\ \left. - 2\lambda\lambda'\right) + \left(\lambda\frac{u_1'}{u_1} + \lambda'^2\right)^2, \end{aligned} \tag{4.10}$$

so its height (which can be estimated counting its possible poles) is bounded by

$$\begin{aligned} H_{\tilde{\mathcal{C}}}(Discr(F(X))) &\leq 2H_{\tilde{\mathcal{C}}}\left(\frac{u_2'}{u_2}\right) + 2H_{\tilde{\mathcal{C}}}\left(\frac{u_1'}{u_1}\right) + 2H_{\tilde{\mathcal{C}}}(\lambda') + 2H_{\tilde{\mathcal{C}}}(\lambda) \\ &\leq 6\chi_S(\tilde{\mathcal{C}}) + 4H_{\tilde{\mathcal{C}}}(\lambda). \end{aligned}$$

Analogously we can look at the discriminant of $G(X)$

$$\begin{aligned} Discr(G(X)) = \left(\frac{u_2'}{u_2}\right)^2 \left[\left(\frac{u_1'}{u_1}\right)^2 \lambda^2(4 - \lambda^2) + \frac{u_1'}{u_1}\lambda\lambda'(8 - 2\lambda^2) + \right. \\ \left. + \lambda'^2(\lambda^2 - 4)\right] + 2\,\frac{u_2'}{u_2}\left[\left(\frac{u_1'}{u_1}\right)^3 \lambda^2(4 - \lambda^2) + \right. \\ \left. + \left(\frac{u_1'}{u_1}\right)^2 \lambda\lambda'(\lambda^2 - 8) + \frac{u_1'}{u_1}\lambda'^2(4 + \lambda^2) - \lambda\lambda'\right] + \\ + \left(\frac{u_1'}{u_1}\right)^4 \lambda^2 - 2\left(\frac{u_1'}{u_1}\right)^2 \lambda^2\lambda'^2 + \lambda'^4 \end{aligned} \tag{4.11}$$

and bound its height in the same way, obtaining that $H_{\tilde{\mathcal{C}}}(Discr(G(X)))$ is bounded above by

$$2H_{\mathcal{C}}\left(\frac{u_2'}{u_2}\right) + 4H_{\tilde{\mathcal{C}}}\left(\frac{u_1'}{u_1}\right) + 4H_{\tilde{\mathcal{C}}}(\lambda') + 4H_{\tilde{\mathcal{C}}}(\lambda) \leq$$
$$\leq 10\chi_S(\tilde{\mathcal{C}}) + 8H_{\tilde{\mathcal{C}}}(\lambda).$$

Therefore the number of ramification points is at most

$$\sharp(S \cup T) + 16\chi_S(\tilde{\mathcal{C}}) + 12H_{\tilde{\mathcal{C}}}(\lambda).$$

We can now apply the Riemann-Hurwitz formula

$$2g(\tilde{\mathcal{D}}) - 2 = (\deg p)(2g(\tilde{\mathcal{C}}) - 2) + \sum_{P \in \tilde{\mathcal{D}}} (e_P - 1). \tag{4.12}$$

Here $e(P)$ is the ramification index of p at P and thus $(e_P - 1)$ is either zero or one. Now we can apply the above estimate of the ramification points of p and we obtain that

$$\sum_{P \in \tilde{\mathcal{D}}} (e_P - 1) \leq \sharp(S \cup T) + 16\chi_S(\tilde{\mathcal{C}}) + 12H_{\tilde{\mathcal{C}}}(\lambda). \tag{4.13}$$

Consider now the set $U \subset \tilde{\mathcal{D}}$ introduced in the statement of the Lemma. We have that

$$\sharp U \leq [\kappa(\tilde{\mathcal{D}}) : p^\star(\kappa(\tilde{\mathcal{C}}))] \cdot \sharp p(U).$$

From this inequality and from (4.12) and (4.13) the quantity $2g(\tilde{\mathcal{D}}) - 2 + \sharp U$ is bounded above by

$$
\begin{aligned}
&(\deg p)\left(2g(\tilde{\mathcal{C}})-2+\sharp p(U)\right)+\sharp(S\cup T)+16\chi_S(\tilde{\mathcal{C}})+12H_{\tilde{\mathcal{C}}}(\lambda)\\
&=(\deg p)\left(2g(\tilde{\mathcal{C}})-2+\sharp(S\cup T)+\sharp(p(U)\setminus(S\cup T)\right)+\\
&\qquad\qquad+\sharp(S\cup T)+16\chi_S(\tilde{\mathcal{C}})+12H_{\tilde{\mathcal{C}}}(\lambda)\\
&\leq 4\chi_{S\cup T}(\tilde{\mathcal{C}})+4\sharp(p(U)\setminus(S\cup T))+\sharp(S\cup T)+16\chi_S(\tilde{\mathcal{C}})+12H_{\tilde{\mathcal{C}}}(\lambda).
\end{aligned}
$$

We have to bound the number $\sharp(p(U)\setminus(S\cup T))$, but the points in the image of U that are not in $S\cup T$ are precisely the zeros of the leading and constant terms in $F(X)$ and $G(X)$. Again we can estimate their number by looking at the height of these terms. We obtain that

$$
\begin{aligned}
\sharp(p(U)\setminus(S\cup T))\leq{}& H_{\tilde{\mathcal{C}}}\left(2\frac{u_1'}{u_1}-\frac{u_2'}{u_2}\right)+H_{\tilde{\mathcal{C}}}\left(\frac{u_2'}{u_2}\right)+\\
&+2H_{\tilde{\mathcal{C}}}\left(2\frac{u_1'}{u_1}-\frac{u_2'}{u_2}\right)+H_{\tilde{\mathcal{C}}}\left(\left(\frac{u_1'}{u_1}\right)^2(4-\lambda^2)+\lambda'^2\right)\\
&\leq 4\chi_S(\tilde{\mathcal{C}})+H_{\tilde{\mathcal{C}}}\left(\left(\frac{u_1'}{u_1}\right)^2(4-\lambda^2)+\lambda'^2\right)\\
&\leq 8\chi_S(\tilde{\mathcal{C}})+4H_{\tilde{\mathcal{C}}}(\lambda).
\end{aligned}
$$

Taking this into account we can return to the previous inequality to finish our proof:

$$
\begin{aligned}
\chi_U(\tilde{\mathcal{D}})\leq{}& 4\chi_{S\cup T}(\tilde{\mathcal{C}})+32\chi_S(\tilde{\mathcal{C}})+16H_{\tilde{\mathcal{C}}}(\lambda)+\sharp S+\sharp T+\\
&+16\chi_S(\tilde{\mathcal{C}})+12H_{\tilde{\mathcal{C}}}(\lambda)\\
&\leq 52\chi_S(\tilde{\mathcal{C}})+28H_{\tilde{\mathcal{C}}}(\lambda)+5\sharp T+\sharp S\\
&\leq 53\chi_S(\tilde{\mathcal{C}})+28H_{\tilde{\mathcal{C}}}(\lambda)+5\max\{0,2g(\tilde{\mathcal{C}})-2\}.
\end{aligned}
$$

□

The next step in the proof of our main result is an application of a theorem by Corvaja and Zannier concerning the "greatest common divisor" of two rational functions on $\tilde{\mathcal{C}}$ of the form $a-1$ and $b-1$ where a and b are units with respect to some specified finite set (in our case the set will be U). In particular we are going to apply Corollary 3.2.9 from page 54 for a suitable choice of units a and b: these units will be chosen in such a way that their heights will be "close" to the heights of u_1, u_2 and such that the sum appearing in the statement of the previous Theorem gives an upper bound for $\sum_{v\in\tilde{\mathcal{D}}\setminus U} v(y)$. We begin by proving the following

Lemma 4.4.5. *Let (u_1, u_2, y) be a solution of equation (4.5) (recall that we are supposing that the leading and constant coefficients of F, G are both non zero). Let $\tilde{\mathcal{D}}, U$ as before. Then there exist U-units $a, b \in \kappa(\tilde{\mathcal{D}})$ such that the quantity*

$$|\max\{H_{\tilde{\mathcal{D}}}(a), H_{\tilde{\mathcal{D}}}(b)\} - \max\{H_{\tilde{\mathcal{D}}}(u_1), H_{\tilde{\mathcal{D}}}(u_2)\}| \qquad (4.14)$$

is bounded above by $32\cdot\chi_S(\tilde{\mathcal{C}}) + 8H_{\tilde{\mathcal{C}}}(\lambda)$ and

$$\sum_{v\in\tilde{\mathcal{D}}\setminus U} \min\{v(1-a), v(1-b)\} \geq \frac{1}{4}\cdot \sum_{v\in\tilde{\mathcal{D}}\setminus U} v(y). \qquad (4.15)$$

Moreover, $a = u_1\alpha^{-1}$, $b = u_2\beta^{-1}$ for suitable α, β such that $F(\alpha) = G(\beta) = 0$.

Proof. Being the field $\kappa(\tilde{\mathcal{D}})$ defined as the splitting field for the polynomial $F(X)\cdot G(X)$ we can write the two polynomials as

$$F(X) = \left(2\frac{u_1'}{u_1} - \frac{u_2'}{u_2}\right)(X-\alpha)\cdot(X-\bar{\alpha}),$$

$$G(X) = \left(2\frac{u_1'}{u_1} - \frac{u_2'}{u_2}\right)^2(X-\beta)\cdot(X-\bar{\beta}).$$

We claim that the roots $\alpha, \bar{\alpha}$ (resp. $\beta, \bar{\beta}$) of F (resp. G) are U-units. This follows from the definition of U (see Lemma

(4.4.4)), because the leading and constant coefficients of the two polynomials are U-units. We consider now the following polynomials obtained from F and G dividing by $\alpha\bar{\alpha}(2\frac{u_1'}{u_1} - \frac{u_2'}{u_2})$ and $\beta\bar{\beta}(2\frac{u_1'}{u_1} - \frac{u_2'}{u_2})^2$ respectively, i.e. the polynomials

$$\overline{F}(X) := (X\alpha^{-1} - 1)(X\bar{\alpha}^{-1} - 1)$$
$$\overline{G}(X) := (X\beta^{-1} - 1)(X\bar{\beta}^{-1} - 1).$$

Now, by Lemma (4.4.3), the U-integer y divides both $F(u_1)$ and $G(u_2)$, and hence it divides the polynomials $\overline{F}(u_1)$ and $\overline{G}(u_2)$ in the ring of U-integers. From this it follows that

$$\sum_{v\in\tilde{\mathcal{D}}\setminus U} \min\{v(u_1\alpha^{-1} - 1) + v(u_1\bar{\alpha}^{-1} - 1), v(u_2\bar{\beta}^{-1} - 1) + v(u_2\bar{\beta}^{-1} - 1)\}$$

is bounded below by $\sum_{v\in\tilde{\mathcal{D}}\setminus U} v(y)$.

We want to analyze the left side term of the last inequality: observe that for every fourple of rational functions W_1, W_2, Z_1, Z_2 one has (we omit the valuations)

$$\sum_v \min\{W_1 + W_2, Z_1 + Z_2\} \le \sum_v \min\{W_1, Z_1\} + \sum_v \min\{W_1, Z_2\} + \\ + \sum_v \min\{W_2, Z_1\} + \sum_v \min\{W_2, Z_2\} \le 4\sum_v \min\{\tilde{W}, \tilde{Z}\},$$

for suitable $\tilde{W} \in \{W_1, W_2\}$ and $\tilde{Z} \in \{Z_1, Z_2\}$. In our case we obtain that there exist U-units $a \in \{u_1\alpha^{-1}, u_1\bar{\alpha}^{-1}\}$ and $b \in \{u_2\beta^{-1}, u_2\bar{\beta}^{-1}\}$ such that:

$$4\sum_{v\in\tilde{\mathcal{D}}\setminus U} \min\{v(a-1), v(b-1)\} \ge \sum_{v\in\tilde{\mathcal{D}}\setminus U} v(y),$$

proving (4.15). Next we want to prove that the heights of these U-units a, b are "close" to the heights of u_1, u_2. We observe that the difference appearing in the left side term of

(4.14) is bounded by the maximum of the $\tilde{\mathcal{D}}$-heights of the roots of F and G. Again we bound these heights by estimating their possible poles. It is then sufficient to observe that the poles of the roots $\alpha, \bar{\alpha}$ (resp. $\beta, \bar{\beta}$) are either zeros of the leading coefficient or poles of the constant term of the polynomial F (resp. G). Hence

$$\begin{aligned}\max\{H_{\tilde{\mathcal{D}}}(\alpha), H_{\tilde{\mathcal{D}}}(\bar{\alpha})\} &\leq H_{\tilde{\mathcal{D}}}\left(2\frac{u_1'}{u_1} - \frac{u_2'}{u_2}\right) + H_{\tilde{\mathcal{D}}}\left(\frac{u_2'}{u_2}\right)\\ &\leq 4H_{\tilde{\mathcal{C}}}\left(2\frac{u_1'}{u_1} - \frac{u_2'}{u_2}\right) + 4H_{\tilde{\mathcal{C}}}\left(\frac{u_2'}{u_2}\right)\\ &\leq 8\chi_s(\tilde{\mathcal{C}}).\end{aligned}$$

In the same way we get the quantity $\max\{H_{\tilde{\mathcal{D}}}(\beta), H_{\tilde{\mathcal{D}}}(\bar{\beta})\}$ is bounded above by

$$\begin{aligned}&H_{\tilde{\mathcal{D}}}\left(2\frac{u_1'}{u_1} - \frac{u_2'}{u_2}\right)^2 + H_{\tilde{\mathcal{D}}}\left[\left(\frac{u_1'}{u_1}\right)^2(4-\lambda^2) + \lambda'^2\right]\\ &\leq 4H_{\tilde{\mathcal{C}}}\left(2\frac{u_1'}{u_1} - \frac{u_2'}{u_2}\right)^2 + 4H_{\tilde{\mathcal{C}}}\left[\left(\frac{u_1'}{u_1}\right)^2(4-\lambda^2) + \lambda'^2\right]\\ &\leq 32\chi_s(\tilde{\mathcal{C}}) + 8H_{\tilde{\mathcal{C}}}(\lambda).\end{aligned}$$

□

In order to apply Theorem (3.2.9) we need an upper bound for $\sum_{v\in\tilde{\mathcal{D}}\setminus U} v(y)$ in terms of the heights of u_1, u_2. This bound is obtained by an application of a theorem by U. Zannier in [Zan] which reads as follows:

Theorem 4.4.6 (Zannier)**.** *Let $\tilde{\mathcal{D}}, U$ as before, $m \geq 2$ an integer, $\theta_1, \ldots, \theta_m$ U-units such that no subsum of $\theta_1 + \cdots + \theta_m$ vanishes. Then the U-integer $\theta_1 + \cdots + \theta_m$ satisfies*

$$\sum_{v\in\tilde{\mathcal{D}}\setminus U} v(\theta_1 + \cdots + \theta_m) \geq H_{\tilde{\mathcal{D}}}(\theta_1 : \cdots : \theta_m) - \binom{m}{2}\chi_U(\tilde{\mathcal{D}}).$$

We are going to apply this Theorem to the U-integer

$$y = u_1^2 + \lambda u_1 + u_2 + 1,$$

using the fact that

$$\begin{aligned} H_{\tilde{\mathcal{D}}}(u_1^2 : \lambda u_1 : u_2 : 1) &\geq \max\{2H_{\tilde{\mathcal{D}}}(u_1), H_{\tilde{\mathcal{D}}}(u_1) + H_{\tilde{\mathcal{D}}}(\lambda), H_{\tilde{\mathcal{D}}}(u_2)\} \\ &\geq \max\{H_{\tilde{\mathcal{D}}}(u_1), H_{\tilde{\mathcal{D}}}(u_2)\}. \end{aligned}$$

In particular, assuming that no subsum of the right term of equation (4.5) vanishes, we obtained the following

Lemma 4.4.7. *For every solution (y, u_1, u_2) of (4.5) such that no subsum of the right term vanishes, one has*

$$H_{\tilde{\mathcal{D}}}(y) \geq \sum_{v \in \tilde{\mathcal{D}} \setminus U} v(y) \geq \max\{H_{\tilde{\mathcal{D}}}(u_1), H_{\tilde{\mathcal{D}}}(u_2)\} - 6\chi_U(\tilde{\mathcal{D}}).$$

Now we put together this last inequality with the results of Lemma (4.14) and we obtain that, for every solution of (4.5) there exist U-units a, b such that the sum

$$\sum_{v \in \tilde{\mathcal{D}} \setminus U} \min\{v(a-1), v(b-1)\}$$

is greater or equal than

$$\frac{1}{4}\left(\max\{H_{\tilde{\mathcal{D}}}(a), H_{\tilde{\mathcal{D}}}(b)\} - 6\chi_U(\tilde{\mathcal{D}}) - 32\chi_S(\tilde{\mathcal{C}}) - 8H_{\tilde{\mathcal{C}}}(\lambda) \right).$$

Using the fact that $\chi_S(\tilde{\mathcal{C}}) \leq \chi_U(\tilde{\mathcal{D}})$ we obtain that the same quantity is bounded below by

$$\frac{1}{4}\left(\max\{H_{\tilde{\mathcal{D}}}(a), H_{\tilde{\mathcal{D}}}(b)\} - 38\chi_S(\tilde{\mathcal{C}}) - 8H_{\tilde{\mathcal{C}}}(\lambda) \right). \tag{4.16}$$

We can now apply Theorem (3.2.9) to deduce the following

Proposition 4.4.8. *Let $(y, u_1, u_2) \in \mathcal{O}_S \times (\mathcal{O}_S^*)^2$ be a solution of equation (4.5) such that no subsum of the right term vanishes, and the leading and constant term of the polynomials F, G are not zero. Let $\tilde{\mathcal{D}}, U$ be as defined in Lemma (4.4.4) and $a, b \in \mathcal{O}_U^*$ as defined in Lemma (4.4.5). Then either*

$$\max\{H_{\tilde{\mathcal{C}}}(u_1), H_{\tilde{\mathcal{C}}}(u_2)\} \leq 2^{12}\left(58\chi_S(\tilde{\mathcal{C}}) + 28H_{\tilde{\mathcal{C}}}(\lambda)\right) + 8H_{\tilde{\mathcal{C}}}(\lambda) \tag{4.17}$$

or a, b verify a multiplicative dependence relation of the form

$$a^r \cdot b^s = 1$$

for integers $(r, s) \in \mathbb{Z}^2 \setminus \{0\}$ with

$$\max\{|r|, |s|\} \leq 5. \tag{4.18}$$

Proof. We suppose that inequality (4.17) does not hold and we want to prove the dependence relation for a, b. In order to apply Corvaja and Zannier Theorem (3.2.9) we are going to show that the left-hand side of (4.17) is greater than the right-hand side of (3.3). Our starting point is

$$\max\{H_{\tilde{\mathcal{C}}}(u_1), H_{\tilde{\mathcal{C}}}(u_2)\} > 2^{12} \cdot \left(58 \cdot \chi_S(\tilde{\mathcal{C}}) + 28H_{\tilde{\mathcal{C}}}(\lambda)\right) + 8H_{\tilde{\mathcal{C}}}(\lambda).$$

From Lemma (4.4.4) we know that

$$\chi_U(\tilde{\mathcal{D}}) \leq 58 \cdot \chi_S(\tilde{\mathcal{C}}) + 28H_{\tilde{\mathcal{C}}}(\lambda)$$

and so we obtain that

$$\max\{H_{\tilde{\mathcal{C}}}(u_1), H_{\tilde{\mathcal{C}}}(u_2)\} > 2^{12}\chi_U(\tilde{\mathcal{D}}) + 8H_{\tilde{\mathcal{C}}}(\lambda).$$

Remember that our aim is to apply Theorem (3.2.9) and so we need to work with the maximum of the heights of a, b.

For this reason we apply (4.14) which estimates the closeness of $H(u_i)$ and $H(a), H(b)$ and, using $H_{\tilde{\mathcal{C}}} \leq H_{\tilde{\mathcal{D}}}$ we get $\max\{H_{\tilde{\mathcal{D}}}(a), H_{\tilde{\mathcal{D}}}(b)\}$ is bounded above by

$$\begin{aligned}&\max\{H_{\tilde{\mathcal{D}}}(u_1), H_{\tilde{\mathcal{D}}}(u_2)\} - 32\chi_S(\tilde{\mathcal{C}}) - 8H_{\tilde{\mathcal{C}}}(\lambda)\\ &\geq \max\{H_{\tilde{\mathcal{D}}}(u_1), H_{\tilde{\mathcal{D}}}(u_2)\} - 32\chi_U(\tilde{\mathcal{D}}) - 8H_{\tilde{\mathcal{C}}}(\lambda).\end{aligned}$$

From these last two inequalities we obtain the lower bound

$$\max\{H_{\tilde{\mathcal{D}}}(a), H_{\tilde{\mathcal{D}}}(b)\} \geq (2^{12} - 32)\chi_U(\tilde{\mathcal{D}}). \tag{4.19}$$

In order to simplify the notation we put $H = \max\{H_{\tilde{\mathcal{D}}}(a), H_{\tilde{\mathcal{D}}}(b)\}$ and $\chi = \chi_U(\tilde{\mathcal{D}})$. We claim that

$$\sum_{v \in \tilde{\mathcal{D}} \setminus U} \min\{v(a-1), v(b-1)\} > 3 \cdot 2^{\frac{1}{3}} H^{\frac{2}{3}} \chi^{\frac{1}{3}} \tag{4.20}$$

To prove the claim we observe that, from (4.16), it is enough to show that

$$\frac{1}{4}H - 38\chi > 3 \cdot 2^{\frac{1}{3}} H^{\frac{2}{3}} \chi^{\frac{1}{3}}.$$

We define the function

$$f(t) = \frac{1}{4}t - 3 \cdot 2^{\frac{1}{3}} t^{\frac{2}{3}} \chi^{\frac{1}{3}} - 38\chi$$

and we notice that our claim is equivalent to $f(H) > 0$. Now the function f is an increasing function for $t \geq 2^{10}\chi$, therefore it is enough to prove it for $H = (2^{12} - 32)\chi > 2^{10}\chi$. Hence the claim is equivalent to

$$\frac{1}{4}(2^{12} - 32)\chi - 3 \cdot 2^{\frac{1}{3}} (2^{12} - 32)^{\frac{2}{3}} \chi > 38\chi$$

With some algebraic manipulations one gets

$$\begin{aligned}
&\frac{1}{4}(2^{12}-32)\chi - 3\cdot 2^{\frac{1}{3}}(2^{12}-32)^{\frac{2}{3}}\chi = \\
&2^{\frac{10}{3}}(2^{7}-1)^{\frac{2}{3}}\left[\frac{1}{4}2^{\frac{5}{3}}((2^{7}-1)^{\frac{1}{3}} - 3\cdot 2^{\frac{1}{3}}\right]\chi = \\
&40\cdot 6\cdot\left[2^{-\frac{1}{3}}\cdot 2^{\frac{13}{6}} - 3\cdot 2^{\frac{1}{3}}\right]\chi = \\
&40\cdot 6\cdot\left[2^{\frac{1}{3}}(2^{\frac{5}{3}}-3)\right]\chi > \\
&40\cdot 6\cdot\left[1\cdot\frac{1}{6}\right]\chi
\end{aligned}$$

which proves the claim. Now we can apply Theorem (3.2.9) which implies that a, b verify a multiplicative dependence relation of the form $a^r b^s = 1$ for some integers r, s not both zero. The same Theorem gives the bound (3.4) and hence, together with (4.19) and (4.16), we obtain

$$\frac{H}{\max\{|r|,|s|\}} > \frac{1}{4}H - 10\chi > \frac{1}{5}H$$

Therefore we get $\max\{|r|,|s|\} \leq 5$, as desired. □

The conclusion of Proposition 4.4.8 gives us a multiplicative relation of dependence between a, b instead of u_1, u_2. However this relation is guaranteed by Lemma 3.14 in [CZ5] which gives us the following result:

Lemma 4.4.9 ([CZ5]). *In the previous notation, if a multiplicative relation of the form $a^r \cdot b^s = \mu$ holds for a constant $\mu \in \kappa$, then either one between a and b is constant or u_1, u_2 satisfy a multiplicative dependence relation of the same type.*

Now we go back to Theorem (4.4.2): here we should take care of the constant term of the polynomial G in a different way as in the constant case. In detail the vanishing of this term does not directly imply an explicit bound for the degree of the images $f(\mathcal{C})$ as in the split function field case; here we should apply again the whole machinery in order to explicitly find the unit u_1 and so reduce the problem to equation $y^2 = \mu + u_2 + 1$, which was already solved in the split case and gives the desired bound. For readability reasons we split the proof of Theorem (4.4.2) in two cases: Lemma (4.4.10) for the case in which the constant coefficient of G is not zero, and Lemma (4.4.11) for the other case. Clearly the two lemmas together gives Theorem (4.4.2).

Lemma 4.4.10. *Suppose that the constant term of the polynomial G does not vanish, i.e., with the notation of 4.4.2, every solution $(y, u_1, u_2) \in \mathcal{O}_S \times (\mathcal{O}_S^*)^2$ of equation (4.5)*

$$y^2 = u_1^2 + \lambda u_1 + u_2 + 1$$

satisfies also

$$\left(\frac{u_1'}{u_1}\right)^2 (4 - \lambda^2) + \lambda^2 \left(\frac{\lambda'}{\lambda}\right)^2 \neq 0. \tag{4.21}$$

Then one of the following conditions holds:

(i) *a sub-sum on the right term of (4.5) vanishes;*

(ii) *u_1, u_2 verify a multiplicative dependence relation of the form $u_1^r \cdot u_2^s = \mu$, where $\mu \in \kappa$ is a scalar and r, s, are integers, non both zeros such that* $\max\{r, s\} \leq 5$;

(iii) *the following bound holds:*

$$\max\{H_{\tilde{\mathcal{C}}}(u_1), H_{\tilde{\mathcal{C}}}(u_2)\} \leq 2^{12} \cdot \left(58 \cdot \chi_S(\tilde{\mathcal{C}}) + 28 H_{\tilde{\mathcal{C}}}(\lambda)\right) + 8 H_{\tilde{\mathcal{C}}}(\lambda).$$

Proof. We start assuming that (i), (ii) and (iii) are not satisfied and we are going to find a contradiction. First of all we note that, if (i) is not satisfied, no subsum of (4.1) can vanish. Moreover the polynomials F and G defined in 4.4.3 could not be constant because the vanishing of their leading coefficients would imply some multiplicative relation between u_1 and u_2 which is excluded by (ii). The same is true for the constant coefficient of F (which is u_2'/u_2): it cannot be zero otherwise u_2

would be constant; moreover, by our assumptions, the same holds for the constant coefficient of G. Hence both F and G are non constant polynomials whose constant coefficients are not zero.

Since we excluded the case where the leading and constant coefficients of F and G vanish, we can apply (4.4.8) and obtain a multiplicative relation between $a = u_1\alpha^{-1}$ and $b = u_1\beta^{-1}$; this follows from the fact that inequality (4.17) is excluded by (iii). From this relation, applying (4.4.9), we get that either a or b is constant or u_1 and u_2 verify a multiplicative relation of the same type. The former case would imply that the height of u_1 (or u_2) would be the same as the height of α (resp. β) so it would be lesser or equal than $8\chi_S(\tilde{\mathcal{C}})$ (resp. $32\chi_S(\tilde{\mathcal{C}})$); but this contradicts our assumption that (iii) does not hold and hence it is excluded. The latter case is precisely (ii) that was assumed to be false. In both cases we get a contradiction and this concludes the proof.

□

Lemma 4.4.11. *Suppose that the constant term of the polynomial G vanishes, i.e., with the notation of 4.4.2, every solution $(y, u_1, u_2) \in \mathcal{O}_S \times (\mathcal{O}_S^*)^2$ of equation (4.5) satisfies also*

$$\left(\frac{u_1'}{u_1}\right)^2 (4-\lambda^2) + \lambda^2 \left(\frac{\lambda'}{\lambda}\right)^2 = 0. \tag{4.22}$$

Then one of the following conditions holds:

(i) (y, u_1, u_2) *satisfy an equation whose solutions verify conclusions of Theorem (4.4.2).*

(ii) u_1, u_2 *verify a multiplicative dependence relation of the form $u_1^r \cdot u_2^s = \mu$, where $\mu \in \kappa$ is a scalar and r, s, are integers, not both zero such that* $\max\{r, s\} \leq 5$*;*

Proof. The first trivial case is the case in which λ is constant which is excluded since we are assuming that the threefold defined by λ is not trivial. The second case is the case in which λ is a non constant S-unit. In this case we obtain, in the ring $\mathcal{O}_S$, the following identity (here we recall that can enlarge S so that it contains every point for which $\lambda = 2$):

$$\left(\frac{u_1'}{u_1}\right)^2 = -\frac{\lambda^2}{4-\lambda^2}\left(\frac{\lambda'}{\lambda}\right)^2. \tag{4.23}$$

Now we observe that the ring $\mathcal{O}_S^*$ is finitely generated modulo constants, so every $u_1 \in \mathcal{O}_S^*$ is of the form $\mu \cdot v_1^{a_1} \cdots v_h^{a_h}$ for some $\mu \in \kappa$ and $v_1, \ldots, v_h \in \mathcal{O}_S^*$. Therefore we have

$$\frac{u_1'}{u_1} = \sum_{i=1}^{h} a_i \frac{v_i'}{v_i}.$$

Being $\lambda \in \mathcal{O}_S^*$ the right-hand side of equation (4.23) could also be expressed in the v_i and their derivatives; in particular 4.23 becomes an equation in the unknown a_i and this equation will have a unique (for given λ and S) solution in the a_i. Hence u_1 will be uniquely determined up to a constant factor and therefore its height will be a constant. So we can assume that $u_1 = af$ for a constant $a \in \kappa$ and a fixed S-unit f. This leads to consider equation $y^2 = a^2 f^2 + \lambda a f + u_2 + 1$. We claim that this case gives (i). The claim follows from a repetition of all the considerations done until now for equation (4.5): we obtain the same estimates with different polynomials $\tilde{F}, \tilde{G}$. Again we look at the vanishing of the constant and leading coefficients and this time we found that the case in which the constant coefficient of the new polynomial $\tilde{G}$ vanishes gives us either $u_1 = 0$ or $u_1 = f$ where $a = 1$. In both cases this reduces the problem to the equation $y^2 = \mu + u_2 + 1$, where μ is now fixed, which has already been treated in the split function field case and gives (i). The case in which the constant term of $\tilde{G}$ is not zero is precisely one of the cases of (i) and this concludes the proof of the claim. □

Finally we prove Theorem (4.1.1) using the previous Theorem.

Proof of Theorem (4.1.1). As in Lemma (4.4.1) a section

$$\sigma : \tilde{\mathcal{C}} \setminus S \to X$$

will be of the form

$$\sigma : P \mapsto (u_1(P), y(P), P)$$

where the S-unit u_1 and the S-integer y verify equation (4.5) for a S-unit u_2. In this setting we can apply Theorem (4.4.2) and conclude that one of (i),(ii),(iii) holds. Let us analyze every case.

- In the first case (i) we have that some sub-sum of $u_1^2 + \lambda(P)u_1 + u_2 + 1$ will vanish. Hence $\sigma(\mathcal{C})$ is either a line or a conic and its degree verifies the bound (recall that being λ non constant its height is at least one).

- In the second case (ii) we have a multiplicative relation between the two S-units of the form $u_1^r = u_2^s \cdot \mu$ for a scalar $\mu \in \kappa$ and two integers r, s with absolute value lesser or equal than 5. From this it follows that $\deg \sigma(\mathcal{C}) \leq H_{\tilde{\mathcal{C}}}(u_1) + H_{\tilde{\mathcal{C}}}(y) \leq 20$ and again the bound is verified.

- In the last case (iii) we have $\max\{H_{\tilde{\mathcal{C}}}(u_1), H_{\tilde{\mathcal{C}}}(u_2)\}$ is bounded above by $2^{12} \cdot \left(58 \cdot \chi_S(\tilde{\mathcal{C}}) + 28 H_{\tilde{\mathcal{C}}}(\lambda)\right) + 8 H_{\tilde{\mathcal{C}}}(\lambda)$ from which we obtain the desired bound.

□

III

The split case

5

Approaches to fewer components

Up to this point we have discussed cases of Lang-Vojta Conjecture in which the number of components of the divisor at infinity D were at least three. The aim of the last part of this Thesis is to develop a strategy to deal with complements of divisor with fewer components, or even irreducible. We begin by giving the following definition:

Definition 5.0.12. *Given a normal crossing divisor D in a smooth projective variety X we call* logarithmic irregularity *of the couple (X, D) the number of irreducible components minus 1.*

Equivalent formulation, similar to the compact case, can be given by defining logarithmic irregularity as the dimension of the space of sections of the logarithmic cotangent bundle.

Previous Theorems like Theorem 3.1.1 and Theorem 3.2.3 and more generally Theorem 3.3.2 imply that, for $\mathbb{P}^2$, the split case of Lang-Vojta Conjecture for function fields is known when the logarithmic irregularity of $\mathbb{P}^2$ and the divisor D is greater or equal to 2, i.e. is greater or equal to the dimension. The first question we want to address is whether this methods can be applied to study the case of logarithmic irregularity strictly smaller than the dimension or not.

5.1 Limits of Corvaja and Zannier methods

The methods applied in chapter 3 and chapter 4, coming from ideas of Corvaja and Zannier, extends to Theorem 3.3.2 at page 56 (and in particular for the projective plane Theorem 3.3.3 at page 56). However it is clear that the key point of all these strategies is, explicitly or implicitly, the existence of a finite map to $\mathbb{G}_m^2$. Such existence follows from the presence of at least three components of the divisor at infinity. Actually the equations of the three components of the divisor D defines such a map as sketched in Remark 3.3.1 at page 56. Once the existence of the map is proved, one constructs an "extra component" coming form the ramification and then reduces the algebraic hyperbolicity to a generalized S-units equation in such a way that the conclusion follows from an application of Brownawell-Masser Theorem (see Subsection 3.3.1 at page 57 for more details).

The question we are interested in now is whether this strategy could be modified to deal with the case of logarithmic irregularity strictly lesser than 2. As discussed above this seems unlikely. Let us focus on a specific key example in which this difficulty is showed:

Example 5.1.1. Let D be a divisor in $\mathbb{P}^2$ consisting of two conics in general position. Let f_1, f_2 be the equations of the two conics. It is defined a natural map from the complement $\mathbb{P}^2 \setminus D$ to $\mathbb{G}_m$ by

$$P \mapsto \frac{f_1(P)}{f_2(P)}.$$

One can also prove that $\mathbb{G}_m$ is the generalized Albanese Variety of $\mathbb{P}^2 \setminus D$ which in particular implies that every map $\mathbb{P}^2 \setminus D$ to the two dimensional torus factors through the previous map to $\mathbb{G}_m$, i.e.

$$\begin{tikzcd} \mathbb{P}^2 \setminus D \arrow[rr] \arrow[dr] & & \mathbb{G}_m^2 \\ & \mathbb{G}_m \arrow[ur] & \end{tikzcd}$$

Thus there is no natural map from $\mathbb{P}^2 \setminus D \to \mathbb{G}_m^2$ allowing to apply Theorem 3.3.3. At the same time once can see how the "algebraic" approach developed in Chapter 4 and Chapter 5 can be applied neither. In facts one could try to mimic the construction of the corresponding equations in S-units and S-integers, but in this case, having only the equations of the two conics defining the complement, one can build up just *one* S-unit and two S-integers. The corresponding equations cannot be handled as the one considered in Chapter 5. As an example of a situation that can occur one could consider the following equation

$$x^2 = P(u, y),$$

in which x, y are S-integers and u is an S-unit for a finite set of points S in a smooth curve $\tilde{C}$ and P is a polynomial. Differentiating with respect to suitable differential form defining on the curve one obtain the following equation

$$2xx' = \frac{u'}{u} \cdot u \frac{\partial P(u,y)}{\partial u} + y' \frac{\partial P(u,y)}{\partial y},$$

where now the term multiplying the derivative with respect to y has height that cannot be bounded independently of y, because y is not a S-unit.

Previous example shows how the case of logarithmic irregularity strictly less than 2 are cannot be recovered from the known cases of bigger irregularity. In particular, the differentiation argument used in the proof of Theorem 3.2.3 and Theorem 4.1.1, that led to an application of a gcd theorem of S-units, does not seems to be recovered here. Namely the lack of a third irreducible component of the divisor D translate into the absence of a natural defined S-unit. This led to consider more difficult Diophantine equations for which, at the present, no general methods can be applied to describe their solutions. At the same time, the same problem arises when one tries to construct a map to the two dimensional algebraic torus, in order to apply Theorem 3.3.2: the equation of a third irreducible component appears to be fundamental for such a map to exists and hence we cannot see any way to recover results for quartics with less than three irreducible components using the constructions and results obtained by Corvaja and Zannier.

We end this section by noticing that such limitation appears also when considering complements of plane curves with higher degree. In fact, Theorem 3.3.3 applies to complements of curves of arbitrary degree, provided that the curve has at least three components and the degree is greater than four. For this reason it is natural to consider other approach for dealing with affine surface with log irregularity lesser than two.

5.2 Ideas from deformations and curve counting

A first systematic treatment of algebraic hyperbolicity for complements of normal crossing divisor in $\mathbb{P}^2$ where the log irregularity is strictly smaller than 2 can be found in Xi Chen's article [Che3] (and previously for surface in the pre-print [Che2]). Although with a different goal in mind, i.e. Kobayashi conjecture on hyperbolicity of complements of nc divisors in $\mathbb{P}^2$, the ideas presented in the papers have some similarities with the approach taken in this Thesis. Therefore we will devote this section to an introduction to Chen's ideas and to its limitation to an extension to degree four divisors.

5.2.1 Chen's argument

The main idea of [Che3], already present in [Che2] is the following: algebraic hyperbolicity for complements $\mathbb{P}^2 \setminus D$ is much easier to prove when D is reducible. Then for a generic D the natural strategy would be to degenerate it to a "highly" reducible divisor, i.e. a union of lines in general position and then try to recover the same result using the knowledge of the case of the union of lines. It turns out that this strategy gives the desired result also in higher dimension, however we are not going to focus on this important aspect of Chen's work and we focus only on the two dimensional case.

In order to state precisely the results let us fix some notation first: in this section S will be a normal projective variety with canonical singularities (although in our application S will always be $\mathbb{P}^2$ hence non-singular), $N_1(S)$ will denote the free abelian group generated by 1-cycles modulo numerical equivalence, and $N^1(S)$ will denote the group of divisor modulo the same equivalence. We call a function $\phi : N_1 \to \mathbb{R}$ an *additive* function on $N_1(S)$ if $\phi \in \mathrm{Hom}(N_1(S), \mathbb{R})$. Since in the case S non singular one has $\mathrm{Hom}(N_1(S), \mathbb{R}) = N^1(S) \otimes \mathbb{R}$, an additive function ϕ corresponds to an $\mathbb{R}$-divisor D and $\phi(\mathcal{C}) = \mathcal{C} \cdot D$.

The first result proved by Chen is a generalization of Theorem 3.1.1 to arbitrary surface and sufficiently reducible divisor.

Theorem 5.2.1 (X. Chen). *Let S be as above, $B = \sum_{i=1}^{n} B_i$ an effective divisor with normal crossing singularities, F a curve on S. Assume that B_i is a very general member of a base point free linear system $\mathbb{P}\mathcal{L}_i$ for each $i \geq 1$, while B_0 and F are fixed curves, meeting properly in smooth points of B_0. Let ϕ be a function*

$$\phi : Z_1 := \{1 - \textit{cycles on } S\} \to \mathbb{R},$$

for which there exists $\epsilon \in \mathbb{R}$ such that

1. *$(K_S + B - B_i) \cdot \mathcal{C} \geq \epsilon\phi(\mathcal{C})$ for all $i \geq 1$ and all $\mathcal{C} \subset S$ non-rigid;*

2. *$2g(C) - 2 + (B - B_0) \cdot C \geq \epsilon\phi(C)$ for all C reduced;*

3. *$\chi_T(C) \geq \epsilon\phi(C)$ for all irreducible components of F, where $T = \nu^{-1}(C \cap B)$ and ν is the normalization of C.*

Then for all reduced irreducible curves C in S, not contained in B the following holds:

$$\chi_T(C) \geq \epsilon\phi(C),$$

where T is as above.

We note that Chen does not use the notion of Euler Characteristic for affine curves as used throughout this Thesis, but rather defines explicitly an intersection multiplicity, that he denotes by $i_S(C, D)$. We used previous notation in order to be consistent with the rest of the Thesis.

Theorem 5.2.1 gives the already known result of algebraic hyperbolicity for the complement of at least 5 lines in $\mathbb{P}^2$: is it sufficient to apply the Theorem with $B_0 = F = \emptyset$, $B - B_0 = L_1, \ldots, L_d$ with $d \geq 5$ and $\phi = \deg$. We stress however that the requirement $d \geq 5$ is crucial since:

$$K_{\mathbb{P}^2} + B - B_i \sim (d-4)H,$$

with H a line in $\mathbb{P}^2$. This feature with be important in the sequel: in particular it gives first evidence that the argument we are going to describe, using Theorem 5.2.1, cannot be applied directly to complements of quartics.

The key idea of Chen's construction is to take Theorem 5.2.1 as a starting point and reducing the proof of algebraic hyperbolicity for complements of irreducible divisor to the Theorem via degeneration. This idea is made explicit in the following

Theorem 5.2.2. *In the notation above, let$\{I_1, I_2, \ldots, I_m\}$ be a partition of $\{1, 2, \ldots, n\}$, $D_0 = B_0$ and let D_k be a very general member of $\mathbb{P}(\otimes_{i \in I_k} \mathcal{L}_i)$ for every $k > 1$. Let D be $D_0 \cup D_1 \cup \cdots \cup D_m$. If $\phi : N_1(S) \to \mathbb{R}$ is an additive function such that there exists a real number ϵ such that*

1. $(K_S + B - B_i) \cdot C \geq \epsilon\phi(C)$ *for all* $i \geq 1$ *and all* $C \subset S$ *non-rigid;*

2. $2g(C) - 2 + (B - B_0) \cdot C \geq \epsilon\phi(C)$ *for all* C *reduced;*

3. $\chi_T(C) \geq \epsilon\phi(C)$ *for each irreducible component* $C \subset F$, *with* $T = \nu^{-1}(C \cap D)$*;*

then for all irreducible curves C *in* S *not contained in* D *the following holds:*

$$\chi_T(C) \geq \epsilon\phi(C).$$

The Theorem formalized the idea of deformations of the divisor D: here the components D_k are degenerated to the union $\cup B_j$ for $j \in I_k$ in such a way that Theorem 5.2.1 can be applied. Choosing as before $S = \mathbb{P}^2$, $B_0 = F = \emptyset$, B_i a line in $\mathbb{P}^2$ for each $i \geq 1$, D a very general curve of degree d in $\mathbb{P}(H^0(\mathcal{O}_S))$ and $\phi = \deg$ one get the following:

Corollary 5.2.3. *Given a very general plane curve* D *of degree* $d \geq 5$, *for all reduced irreducible curves* $C \subset \mathbb{P}^2$ *not contained in* D, *denoted by* $T = \nu^{-1}(C \cdot D)$, *with* ν *the normalization of* C, *the following holds:*

$$\chi_T(C) = 2g(C) - 2 + \#T \geq (d-4)\deg C.$$

In particular for all very general plane curves D *of degree* $d \geq 5$, $\mathbb{P}^2 \setminus D$ *is algebraic hyperbolic.*

We note again that the bound $degD \geq 5$ is strict since both in Theorem 5.2.2 and in Corollary 5.2.3 the statement and the proof will be false if we consider D to be a generic quartic. Moreover, in Chen's paper, his definition of algebraic hyperbolicity refers to what we called *strong* algebraic hyperbolicity (see page 41); this in particular implies that Theorem 5.2.2 gives that no map $\mathbb{G}_m \to \mathbb{P}^2 \setminus D$ exists provided that $\deg D \geq 5$ and D is a very general member of $\mathcal{O}_{\mathbb{P}^2}(d)$.

Idea of the Proof of Theorem 5.2.2 As pointed out above the main idea of the proof is to degenerate D to a union $\cup B_i$

for which Theorem 5.2.1 can be applied. What actually is done is to degenerate one component of D at the time, while keeping the remaining fixed, and then applying an induction argument on the number of irreducible components.

Let $X = S \times \Delta$, with Δ the unit disk parametrized by t, and let W be an effective divisor in the family X such that

- W is the sum of m components $W^{(1)}, \dots, W^{(k)}$;
- for each $k \neq 1$, $W^{(k)}$ restricted to each fiber is D_k;
- $W^{(1)}$ is a pencil of curves in $\mathbb{P}(\otimes_{i \in I_1} \mathcal{L}_i)$ where the central fiber $W_0^{(1)} = B_1 \cup D_1'$ is reducible, while the general fiber is a general element of the linear series.

Basically the family consider the case in which $D_2, \dots, D_m$ are fixed D_1 moves in a family $W^{(1)}$ in which the general fiber W_t^1 is a general member of the linear series $\mathbb{P}(\otimes_{I_1} \mathcal{L})$, and the special fiber over $t = 0$ reduces to a union $B_1 \cup D_1'$. In this setting one can consider a family of curves $Y \to \Delta$, flat over Δ, with a proper map $\pi : Y \to X$ and a commutative diagram

$$\begin{tikzcd} Y \arrow[r, "\pi"] \arrow[d] & X \arrow[d] \\ \Delta \arrow[r, "\overline{\alpha}"] & \Delta \end{tikzcd}$$

with $\overline{\alpha}$ a base change of order α. Up to applying semistable reduction one can assume that:

- Y is smooth and irreducible and Y_0 is nodal;
- $\pi|_{Y_t}$ maps Y_t birationally onto its image for all $t \neq 0$;
- $\pi(Y)$ meets W properly;
- $Y_t \cap \pi^{-1}(W)$ extends to disjoint sections of $Y \to \Delta$;
- $Y \to X$ is minimal with the above properties.

The conclusion of the Theorem is equivalent to prove:

$$\chi_T(Y_t) \geq \epsilon \phi(\pi(Y_t))$$

for $t \neq 0$ assuming that

$$\chi_{T_0}(\mathcal{C}) \geq \epsilon\phi(\mathcal{C}),$$

with $\mathcal{C} \subset S$ irreducible not contained in W_0 and where T is the set of inverse images in the normalization of Y_t of the intersection with W_t and T_0 are the inverse image, in the normalization of $\mathcal{C}$, of the intersection with W_0. Now one can define, for each component Γ of Y_0, a finite set of points $\sigma(\Gamma)$ which counts the point of intersection in Γ coming from

$$\lim_{t\to 0} Y_t \cap \pi^{-1}(W)$$

and the nodes of Y in Γ. Then the following holds:

$$\chi_T(Y_t) = \sum_{\Gamma}(2p_a(\Gamma) - 2 + \#\sigma(\Gamma)).$$

Then one is reduced to show that

$$2p_a(\Gamma) - 2 + \#\sigma(\Gamma) \geq \epsilon\phi(\pi(\Gamma)).$$

At this point Chen studies separately the case in which Γ is contracted, $\pi(\Gamma)$ intersect properly W_0 and finally, the most delicate case, when the support of $\pi(\Gamma)$ is contained in W_0. In both cases one can recover the desired bound for the Euler Characteristic determining the set $\sigma(\Gamma)$ via a subtle blowup construction and a lower bound for the ramification of the reduction map of the normalization of each component Γ.

5.2.2 Limits of Chen methods

We end this Chapter by a brief analysis of the features of Chen's result and proof that will resemble part of ideas developed in the next chapter.

First of all the most difficult step in the above proof is to deal with curves, or components of curves, that are mapped to a component of the divisor W_0. In facts, as described in Chen's paper, one can look at the family $\pi : Y \to X$ as a family of pre-stable maps with marked points given by the intersection with the divisor W. With this picture in mind one should deal with the possible degeneration that the image of such family of maps could acquire in the fiber over $t = 0$. This is the hard part in which a careful description and control of the set $\sigma(\Gamma)$ plays a fundamental role. The use of pre-stable map however suggests that some moduli space of pre-stable maps could enter the picture allowing a more abstract reformulation of the problem of hyperbolicity. This will be the path we are going to follow in the last part of this Thesis.

Secondly we note how the deformation argument sketched above is really natural when dealing with hyperbolicity questions; Namely, in many cases (if not all the known cases), a result for complements of reducible divisor, usually union of hyperplanes in generic position, is already available, or can be recovered in an easier way. Using this knowledge one could try to obtain a result for a generic irreducible divisor degenerating it to a union of sufficiently many components having normal crossing and recover the desired conclusion from the known results in the reducible case. However this strategy works only for algebraic hyperbolicity and cannot be extended to Kobayashi hyperbolicity, which was the starting point of Chen's analysis: unfortunately Kobayashi Hyperbolicity is not an algebraic condition on varieties.

Thirdly we want to stress the fact that the difficulty in dealing with the set σ as well as with the set S used in previous chapter, is that its cardinality depends only on the *number* of points of intersection but not on the *multiplicities* of intersection. Moreover, for non ordinary multiple points, the multiplicity coming from the singularity is not detected either. Hence one can say that the major difficulty in the previous construction is that, when dealing with the family of pre-stable maps $Y \to X$, there is no control on the multiplicities of intersection in the family.

Finally, from the fact that Chen was interested in *strong* algebraic hyperbolicity and not on log-general type surface, part of its method cannot simply be extended to deal with complements of very generic quartics in $\mathbb{P}^2$. In particular his stronger result for complements of very generic curves of degree at least 5, together with its generalization to higher dimension, provides a strategy to prove that there are no G_m immersion on the varieties considered, a result that we know to be false in a generic log-general type surface (see Remark 2.2.7 at page 42). Moreover, all the results in its paper requires a very generic hypothesis that makes such results weaker than the corresponding one obtained by Corvaja and Zannier in the three component case.

From these observation the idea of the present work is to take a different approach to this problem via Logarithmic Geometry. This will allow to control in an efficient way the multiplicities of intersection, incorporating them as a part of the data of the deformation. Namely Logarithmic Geometry will give us tools that consent to fix such multiplicities in a family of stable maps, once we consider the family as a family of logarithmic maps. Then we will, in some sense, extend Chen's deformation arguments for study algebraic hyperbolicity, in the weak sense, for complements of very generic quartic in $\mathbb{P}^2$. The role played by Theorem 5.2.1 will be taken by Corvaja and Zannier's Theorem 3.2.3. Despite the naturality of a logarithmic approach, we will need a heavy machinery coming form the theory of Logarithmic Stable maps to Deligne-Falting pairs, as developed by Qile Chen and Dan Abramovich. We will devote all the next chapter to recall the basic facts we need for state the mail result for complements of quartics.

6

Logarithmic Geometry

Logarithmic Geometry can be traced back to ideas of Fontaine and Illusie; the first systematic discussion of its foundations has been given by Kazuya Kato in [Kat2]. The theory has been applied with success both in Arithmetic Geometry (works by, among others, Faltings, Kato and Tsuji) and in Algebraic Geometry (related to, Toric Geometry, (p-adic) Hodge Theory, Moduli Spaces and others). In this chapter we will recall the basic definitions needed for fully describing the ideas of the next chapter. For more detailed and extensive introduction to this fascinating theory we refer the reader to Arthur Ogus lectures [Ogu], to the first sections of [ACG$^+$] and to [Ill]. This chapter is organized as follows: the first section will recall the basic notion of logarithmic structure, logarithmic schemes and maps moving from the pivotal example of a normal crossing divisor in a smooth variety. In the second section the focus will be the notion of logarithmic curve which is the natural generalization of pre-stable curve. We are going to briefly describe the notion of stable logarithmic curve giving evidence that this notion naturally generalizes usual stable curves: in particular we will show that log curves encodes naturally a marked structure as well as some constrains on the possible singularities. Then we will see how it is possible to give e logarithmic structure to any pre-stable curve in a canonical way.

The last section will be dedicated to log stable maps, a generalization and an extension of usual stable maps originated from the work of Kontsevich [Kon]. In this direction we will concentrate on the moduli of stable maps and its analogous in the logarithmic category, the stack of log stable maps in the particular case in which the target log scheme is a Deligne Falting pair, a structure to which belongs the couple we are mostly interested in, i.e. $\mathbb{P}^2$ and a reduced simple normal crossing divisor D.

In the context of this Thesis the importance of Logarithmic Geometry, and more in detail of log stable maps and their moduli, comes from the fact that they provide the appropriate framework in which Theorem 3.2.3 can be extended. Actually, the reformulation and extension of Corvaja and Zannier's Theorem in terms of Logarithmic Geometry will be the key point that will allow to extend the result to complements of more general divisors.

6.1 Background and basic constructions

Throughout this Thesis we will assume that all monoids, i.e. semi-groups with unit, are commutative and all morphisms between monoids preserve the unit elements. The structure sheaf of a scheme will be viewed as a sheaf of monoids under multiplication, unless otherwise specified.

We begin by the motivating example of a normal crossing divisor in a smooth variety: this is not only one of the main historically motivation of Logarithmic Geometry, but it is also the one that we will use the most in the following parts of this Thesis.

Example 6.1.1. Let X be a smooth variety (or more generally a regular scheme) and let D be a normal crossing divisor on X. If U denotes the complement $X \setminus D$, one can try to describe the invertible elements of the ring $\mathcal{O}_X(U)$, i.e. regular function of X that are invertible when restricted to U. In a more functorial way one can try to define a sheaf $\mathcal{M}$ on X that associates to each open subset V of X the set $\mathcal{M}(V)$ of regular functions of X whose restriction to $V \setminus D$ is invertible, i.e.

$$\mathcal{M}(V) = \{f \in \mathcal{O}_X(V) : f|_{V \setminus D} \in \mathcal{O}_X(V \setminus D)^*\}.$$

Then the following facts hold:

1. $\mathcal{M}$ can be given the structure of sheaf of monoids on X: it is clear that $\mathcal{M}$ defines a sheaf, however sum of invertible functions needs not to be invertible and hence there is no ring-structure on $\mathcal{M}(V)$ for every V.

2. $\mathcal{M}$ contains the sheaf of units $\mathcal{O}_X^*$. In particular there exists a well-defined map $\mathcal{M} \to \mathcal{O}_X$ which is the identity when restricted to $\mathcal{O}_X^*$.

3. Informally $\mathcal{M}$ "remembers" the inclusion $U \to X$: while being defined on X encodes information about U and his complement D.

4. Derivatives of sections of $\mathcal{M}$ generate the space of sections of the sheaf of differentials $\Omega_X^1(D)$ with logarithmic poles along D which (partially) justifies the terminology of *logarithmic* geometry.

5. The normal crossing condition can be rephrased in local terms if one considers the étale topology instead of the Zariski topology.

6.1.1 Log structure and log-schemes

Motivated by the previous example we give the following definition due to K. Kato

Definition 6.1.2. *Given a scheme X a* pre-logarithmic structure *on X is a sheaf of monoid $\mathcal{M}$ on the étale site of X, together with a morphism of sheaves of monoids $\alpha : \mathcal{M} \to \mathcal{O}_X$ where $\mathcal{O}_X$*

is viewed as a sheaf of monoids under multiplication. The map α is sometimes called the structure morphism.

If the map $\alpha^{-1}\mathcal{O}_X^* \to \mathcal{O}_X^*$ given by α is an isomorphism, $\mathcal{M}$, or rather the couple $(\mathcal{M}, \alpha)$, is called a logarithmic structure *on the scheme X.*

Remark 6.1.3.

1. The sheaf $\mathcal{M}$ defined in example 6.1.1 is a logarithmic structure on the smooth variety X with the inclusion map $\mathcal{M} \to \mathcal{O}_X$.

2. There is an associated logarithmic structure to every pre-logarithmic structure $(\mathcal{M}, \alpha)$ indicated by $\mathcal{M}^a$: is given by
$$\mathcal{M}^a = \mathcal{O}_X^* \oplus_{\alpha^{-1}\mathcal{O}_X^*} \mathcal{M},$$
i.e. the quotient (in the category of monoids) of $\mathcal{O}_X^* \oplus \mathcal{M}$ by the relation on sections over a geometric point $P \in X$ defined by
$$(u, a) \sim (u', a') \Leftrightarrow \exists v, v' \in \alpha^{-1}\mathcal{O}_{X,P}^* \ : \begin{cases} u \cdot \alpha(v') = u' \cdot \alpha(v) \\ v + a = v' + a' \end{cases}$$
(this characterization is possible because $\mathcal{O}_X^*$ is actually a group under multiplication; see [Kat2] (1.3)). Equivalently the associated logarithmic structure is obtained by the pushout of the following diagram in the category of monoids over the étale site
$$\begin{tikzcd} \alpha^{-1}\mathcal{O}_X^* \arrow[r, "\alpha"] \arrow[d] & \mathcal{M} \ . \\ \mathcal{O}_X^* & \end{tikzcd}$$
The structure map for the associated logarithmic structure $\mathcal{M}^a$ is given by
$$\mathcal{M}^a \to \mathcal{O}_X \qquad (u, a) \mapsto u \cdot \alpha(a).$$

3. Given a monoid M and a ring R there is a canonical logarithmic structure that can be defined on $X = Spec\, R[M]$,

where $R[M]$ denotes the monoid algebra (the R-algebra whose underlying module structure is free with basis M). The log structure comes from the canonical map $M \to R[M]$. Usually the scheme $X = Spec\, R$ with its canonical logarithmic structure is denoted by $Spec\,(M \to R[M])$.

Definition 6.1.4. *A* logarithmic scheme, *usually called a* log scheme, *is a scheme X together with a log structure $(\mathcal{M}, \alpha)$ defined on X. We usually denote the log scheme by X when the logarithmic structure is clear, or by the couple $(X, \mathcal{M})$, suppressing the map α. If we need to specify the underlying scheme rather than the log scheme we will use the standard notation (after [Ogu] and [ACG$^+$]) $\underline{X}$.*

Definition 6.1.5. *Given a log scheme $(X, \mathcal{M}_X)$, the* characteristic *of the log structure $\mathcal{M}_X$ is the quotient sheaf*

$$\overline{\mathcal{M}} := \mathcal{M}_X / \mathcal{O}_X^*.$$

In the same way we focus on varieties rather than on arbitrary schemes, we are going to describe a smaller class of logarithmic schemes where the underlying sheaves of monoids possess nice properties that, quoting Kato, are not "too pathological". We begin by introducing properties of log structures analogous to being quasi-coherent, coherent and locally free for sheaves.

Definition 6.1.6. *A log-structure $\mathcal{M}$ on a scheme X is called* quasi-coherent *if étale locally there exists a monoid P together with a homomorphism $\mathcal{P}_X \to \mathcal{O}_X$, where $\mathcal{P}_X$ denotes the constant sheaf associated to P, such that the associated logarithmic structure is isomorphic to $\mathcal{M}$. If the monoid P can be chosen to be finitely-generated then $\mathcal{M}$ is called a* coherent *log structure on X. If $P \simeq \mathbb{N}^k$ for some k, the log structure $\mathcal{M}$ is called* locally free.

In the same spirit, seeking an analogous for normal schemes, we give the following definitions:

Definition 6.1.7. *A monoid $(M, +)$ is called* integral *if, for every $m_1, m_2, m \in M$ such that $m_1 + m = m_2 + m$ it holds $m_1 =$*

m_2. This is equivalent to require that the natural map $M \to M^{gp}$ is injective, where M^{gp} is the associated group of the monoid M.

A monoid $(M, +)$ is called saturated *if it is integral and for every $m \in M^{gp}$, $n \in N$, if $n \cdot m \in M$ then $m \in M$.*

Definition 6.1.8. *A log structure $\mathcal{M}$ is called* integral *if it is a sheaf of integral monoids. It is called* fine *if it is coherent and integral; is called* fine and saturated*, or more briefly* fs*, if it is a coherent sheaf of saturated monoids.*

Remark 6.1.9. Integral quasi-coherent log structures can be characterized as follows: given a quasi-coherent log structure $\mathcal{M}$ on a scheme X, $\mathcal{M}$ is integral if étale locally is isomorphic to the log structure associated to the pre log structure $\mathcal{P}_X \to \mathcal{O}_X$ for some integral monoid P. The same holds for fine log structures replacing *quasi-coherent* with *coherent* and *integral* with *finitely generated and integral* (see Definition 6.1.16 for an equivalent reformulation using charts).

Example 6.1.10. 1. In the setting of example 6.1.1 let $\mathcal{M}_D$ be the log structure induced by the normal crossing divisor D. By the normal crossing hypothesis, the divisor D can be written (étale) locally as the union of regular closed subschemes given by $f_i = 0$ with $i = 1, \dots, r$ for some positive integer r. In this notation, the log structure $\mathcal{M}_D$ is the log structure associated to the pre log structure given by

$$\mathbb{N}^r \to \mathcal{O}_X \qquad (n_1, \dots, n_r) \mapsto \prod_i f_i^{n_i}.$$

In particular $\mathcal{M}_D$ is fine and saturated, being $\mathbb{N}$ finitely generated, integral and saturated.

2. To an integral monoid P with no invertible element other than the unit one can associate an integral logarithmic structure $\mathcal{M}$ on a scheme X via $\mathcal{M} := \mathcal{O}_X^* \oplus P \to \mathcal{O}_X$ defined by

$$(u, a) \mapsto \begin{cases} u & \text{if } a = 1 \\ 0 & \text{if } a \neq 1 \end{cases}$$

It can be shown that every integral logarithmic structure arise in this way. An important example is the so-called *standard logarithmic point* defined in the following way: let $X = Spec\, k$ be the 0-dimensional affine scheme for a field k, and consider the integral log structure on X associated to the integral monoid $\mathbb{N}$ as above. Such a structure, with the additive notation for $\mathbb{N}$ is given by

$$k^* \oplus \mathbb{N} \to \mathcal{O}_X \qquad\qquad (u,n) \mapsto \begin{cases} u & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{cases}$$

and is usually denoted by

$$(u,n) \mapsto u \cdot 0^n,$$

where one defines $0^0 = 1$ and $0^n = 0$ if $n \neq 0$.

6.1.2 Log maps

Definition 6.1.11. *Given two log structures $\mathcal{M}_1$ and $\mathcal{M}_2$ on a scheme X, a* morphism *of log structure is a morphism of sheaves of monoid $\mathcal{M}_1 \to \mathcal{M}_2$ compatible with structure morphisms, i.e. such that the following diagram is commutative*

$$\begin{tikzcd} \mathcal{M}_1 \arrow[rr] \arrow[dr, "\alpha_1"'] & & \mathcal{M}_2 \arrow[dl, "\alpha_2"] \\ & \mathcal{O}_X & \end{tikzcd} .$$

Definition 6.1.12. *Given a morphism of schemes $f : X \to Y$ and a (pre) log structure $\mathcal{M}$ on Y, the* inverse image *of $\mathcal{M}$, denoted by $f^*\mathcal{M}$ is the logarithmic structure associated to the pre log structure on X associate to the map*

$$f^{-1}(\mathcal{M}) \to f^{-1}\mathcal{O}_Y \to \mathcal{O}_X.$$

Example 6.1.13. 1. The canonical log structure defined in Remark 6.1.3 on $Spec\, R[M]$ for a monoid M and a ring R

associated to the map $M \to R[M]$ can be defined equivalently as the pullback of the canonical log structure on $Spec\,(M \to \mathbb{Z}[M])$ via the map $\mathbb{Z} \to R$.

2. The log structure of the standard logarithmic point defined in 6.1.10 can be seen as the inverse image of the log structure on $Spec\, k[x_1, \dots, x_n]$ given by the divisor D : $x_i = 0$ for some i, via the map $Spec\, k \to Spec\, k[x_1, \dots, x_n]$ that sends the point $Spec\, k$ to the origin of the affine space $Spec\, k[x_1, \dots, x_n]$.

With these two notions one can define maps in the category **LSch** of log schemes in the following way:

Definition 6.1.14. *A morphism of log schemes $(X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ is a couple $(f, f^\flat)$ where $f : \underline{X} \to \underline{Y}$ is a morphisms in the category of schemes and $f^\flat : f^*\mathcal{M}_Y \to \mathcal{M}_X$ is a morphism of log structures on X.*

In the same way characteristics were defined for schemes we can define the analogous notion for maps:

Definition 6.1.15. *Given a log morphism $f : (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ the* characteristic *of f, or the* relative characteristic, *is the quotient monoid sheaf $\mathcal{M}_X/\mathcal{N}$ where $\mathcal{N}$ is the image of $f^*\mathcal{M}_Y \to \mathcal{M}_X$.*

In order to work locally on log schemes and log maps, we introduce the useful notion of chart:

Definition 6.1.16. *Given a log scheme $(X, \mathcal{M}_X)$, and a monoid P, a* chart *for $\mathcal{M}_X$ is a morphism $P \to \Gamma(X, \mathcal{M}_X)$ (or equivalently $\mathcal{P}_X \to \mathcal{M}_X$) such that the induced map on logarithmic structure $\mathcal{P} \to \mathcal{M}$ is an isomorphism, where $\mathcal{P}$ denotes the log structure associated to the prelog structure induced by the chart, i.e. to $P \to \Gamma(X, \mathcal{M}_X) \to \Gamma(X, \mathcal{O}_X)$.*

Remark 6.1.17. Give a map $P \to \Gamma(X, \mathcal{M}_X)$ is equivalent to give a map of log schemes

$$(X, \mathcal{M}_X) \to Spec\,(P \to \mathbb{Z}[P]),$$

see [Ogu] for more details.

Having defined charts for log structures we can make more precise Definition 6.1.8 in the following way: given a fine log scheme $(X, \mathcal{M}_X)$ from the coherence of $\mathcal{M}_X$ we get a map $\mathcal{P}_X \to \mathcal{O}_X$ from the constant sheaf associated to a finitely generated monoid P whose associated log structure is isomorphic to $\mathcal{M}_X$. This map induces a map $P \to \Gamma(X, \mathcal{M}_X)$ which is a chart for $\mathcal{M}_X$. This shows that for fine log schemes charts always exists. In particular fine, fine and saturated and locally free log schemes can be characterized using charts.

The notion of chart can be extended to morphisms between log schemes in a natural way:

Definition 6.1.18. *Given a morphism of log schemes*

$$f : (X, \mathcal{M}) \to (Y, \mathcal{N}),$$

a chart *for f is a triple*

$$(\mathcal{P}_X \to \mathcal{M}, \mathcal{Q}_Y \to \mathcal{N}, Q \to P)$$

consisting of

1. *a chart $\mathcal{P}_X \to \mathcal{M}$ for $(X, \mathcal{M})$ and a monoid P;*
2. *a chart $\mathcal{Q}_Y \to \mathcal{N}$ for $(Y, \mathcal{N})$ and a monoid Q;*
3. *a monoid homomorphism $Q \to P$.*

such that the following diagram is commutative:

$$\begin{tikzcd}
\mathcal{Q}_X \arrow[r] \arrow[d] & \mathcal{P}_X \arrow[d] \\
f^*\mathcal{N} \arrow[r] & \mathcal{M}
\end{tikzcd}$$

Remark 6.1.19. Similarly to the case of charts of log structure, if the schemes $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ are fine log schemes, a chart for the map $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ always exists (see [Kat2], Lemma 2.10).

6.1.3 Log smoothness

In the following subsection we are going to discuss the notion of smoothness in the logarithmic category: one of the main point we are going to focus on, is that log smoothness will not a priori implies flatness of the underlying morphism. In this setting the notion of integral log structure will provide the extra condition needed for such a flatness to hold.

We begin by the following definition:

Definition 6.1.20. *Given a morphism of fine log schemes $f : (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$, f is called* log(arithmically) smooth *if the underlying morphism between schemes $f : X \to Y$ is locally of finite presentation, i.e. affine locally represented by a map*

$$Spec\, A \to Spec\, B \qquad where \qquad A = \frac{B[x_1, \ldots x_n]}{f_1, \ldots, f_r}$$

and for any commutative diagram of log schemes

$$\begin{tikzcd} T_0 \arrow[r, "\phi"] \arrow[d, "j"] & X \arrow[d, "f"] \\ T_1 \arrow[r, "\psi"'] & Y \end{tikzcd} ,$$

where j is a (strict) closed immersion defined by a square zero ideal and ϕ, ψ are log morphisms, there exists a morphism $g : T_1 \to X$ such that $\phi = g \circ j$ and $\psi = f \circ g$, i.e. such that the following diagram commutes

$$\begin{tikzcd} T_0 \arrow[r, "\phi"] \arrow[d, "j"] & X \arrow[d, "f"] \\ T_1 \arrow[r, "\psi"'] \arrow[ur, "g"] & Y \end{tikzcd} .$$

This is equivalent to require the map f to be formally smooth *and locally of finite presentation in the category of log schemes.*

Logarithmic smoothness is related to the smoothness of the underlying map between schemes in the following way:

Proposition 6.1.21. *Let $f : (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ be a log morphisms between fine log schemes. If $f^*\mathcal{M}_Y \simeq \mathcal{M}_X$ then f is log smooth if and only if f is smooth as a map between the underlying schemes.*

Note that a log-smooth map needs not to be smooth as a map between the underlying schemes. The typical example is the map from a curve with only normal crossing singularities to a log point: one can see that the map is log smooth while the underlying map between schemes has singularities being the curve singular. As an explicit example one can consider the map

$$f : Spec\, \frac{\mathbb{C}[x_1, x_2]}{x_1 \cdot x_2} \to Spec\, (\mathbb{N} \to \mathbb{C}),$$

where the first log structure is given by

$$\mathbb{N}^2 \to \frac{\mathbb{C}[x_1, x_2]}{x_1 \cdot x_2},$$

sending $e_i \mapsto x_i$, and the log map is defined by

$$\begin{tikzcd} \mathbb{N}^2 \arrow[r] & \frac{\mathbb{C}[x_1, x_2]}{x_1 \cdot x_2} \\ \mathbb{N} \arrow[u, "\Delta"] \arrow[r] & \mathbb{C} \arrow[u] \end{tikzcd},$$

with Δ the diagonal. This is another point in which Log Geometry allows to deal with singularity as if they were smooth.

As mentioned above log smoothness does not imply flatness of the underlying map but if we require the map to be integral we get the flatness. Let us first recall the following

Definition 6.1.22. *A morphism $f : (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ of fine log schemes is called* integral *if for every closed point $x \in X$ the map $f^{-1}(\overline{\mathcal{M}}_Y)_x \to \overline{\mathcal{M}}_{X,x}$ induces a flat map on the monoid algebras $\mathbb{Z}[f^{-1}(\overline{\mathcal{M}}_Y)_x] \to \mathbb{Z}[\overline{\mathcal{M}}_{X,x}]$.*

With the integrality assumptions flatness follows from log smoothness. The following theorem holds:

Theorem 6.1.23 (Kato). *Given a log smooth and integral morphism between fine logarithmic schemes, the underlying map of schemes is flat.*

6.2 Log stable curves

Logarithmic geometry provides tools to extend the usual notion of stable curve and stable maps to the relative case. For curves the functor of the moduli problem for genus g, n marked stable curves is representable by a DM stack $\mathcal{M}_{g,n}$ which compactifies the open stack of smooth irreducible curves of genus g with n markings; moreover the boundary given by singular stable curves is a normal crossing divisor $\mathcal{D}$. Therefore it is given a natural logarithmic structure $\mathcal{M}_{\mathcal{M}}$ on the moduli spaces $\mathcal{M}_{g,n}$. Analogous construction, using inverse image of $\mathcal{M}_{\mathcal{M}}$ and the divisors defined by sections, can be made on the universal family $\mathfrak{C}_{g,n}$ in a canonical way. One then expects that each stable curve can inherit a logarithmic structure from these canonical logarithmic structures. This is the starting point for the definition of log stable curve.

6.2.1 Log curves as stable curves

Recall that a genus g, n pointed pre-stable curve over a scheme S is a flat projective morphism of schemes $\mathcal{C} \to S$ with n sections such that every geometric fiber is a projective, connected, reduced, (at most) nodal curve of arithmetic genus g; it is called a *stable* curve if its group of automorphisms is finite. Equivalently, the stability condition can be expressed by requiring that the twisted dualizing sheaf $\omega_{\mathcal{C}_s}(P_1 + \cdots + P_n)$ is ample, where $\mathcal{C}_s$ is a geometric fiber and $P_1, \ldots, P_n$ are its marked points. Our first goal is to show that stable genus g n

pointed curves and log curves are, in some sense, "faces of the same coin", i.e. on one side to each genus g, n pointed stable curve can be given a log structure in a natural way, and on the other side, each log curve has a structure of a usual pre-stable curve where the log structure encodes information similar to the pointed structure. These results were first established by Fumiharu Kato in [Kat1]; we refer to this paper, together with [ACG$^+$], for details and complete proofs.

We begin with the following

Definition 6.2.1 ([Kat1] 1.2)**.** *A* log curve *over an fs log scheme $(S, \mathcal{M}_S)$, is a log smooth and integral morphism $f : X \to S$ of fs log schemes such that each geometric fiber is a connected and reduced curve.*

We note that from the integrality assumption it follows that the map f is flat as a map between the underlying schemes. In order to deal with moduli space we specify the definition of arrows with the following

Definition 6.2.2. *An* isomorphism *of log curves $\mathcal{C}_1 \to S$ and $\mathcal{C}_2 \to S$ over an fs scheme S with log structure $\mathcal{M}_{S_1}, \mathcal{M}_{S_2}$ respectively, is a couple (σ, γ) where*

1. *$\sigma : (S, \mathcal{M}_{S_1}) \to (S, \mathcal{M}_{S_2})$ is an isomorphism of log schemes such that the underlying map between schemes is the identity;*
2. *$\gamma : (\mathcal{C}_1, \mathcal{M}_1) \to (\mathcal{C}_2, \mathcal{M}_2)$ is an isomorphisms of log schemes whose underlying map between schemes is an isomorphism of pre-stable curves over $\underline{S}$;*
3. *the two maps are compatible with the log curve structure, i.e. the following diagram is commutative*

$$\begin{tikzcd} (\mathcal{C}_1, \mathcal{M}_1) \arrow[r] \arrow[d] & (\mathcal{C}_2, \mathcal{M}_2) \arrow[d] \\ (S, \mathcal{M}_{S_1}) \arrow[r] & (S, \mathcal{M}_{S_2}) \end{tikzcd}$$

With the definition of arrows we can form a category of log curves together with a stability condition that we will describe later. The main feature of log curves is that the fs hypothesis together with the log-smoothness restrict the possible singularities that can appear in such curves. The following theorem holds:

Theorem 6.2.3 ([Kat1] 1.3)**.** *Given a log curve*

$$f : (X, \mathcal{M}_X) \to (S = Spec\, k, \mathcal{M}_S)$$

with k separable closed then X has at worst nodal singularities. Moreover, call $r_1, \ldots, r_l$ the set of double points of X, then there exists $s_1, \ldots, s_n$ points distinct from the nodes, such that the characteristic of f is given by

$$\overline{\mathcal{M}}_{X/S} = \mathbb{Z}_{r_1} \oplus \cdots \oplus \mathbb{Z}_{r_l} \oplus \mathbb{N}_{s_1} \oplus \cdots \oplus \mathbb{N}_{s_n},$$

where for a monoid P we denote by P_x the skyscraper sheaf supported at x.

As a corollary we see that, not only the log curve structure on $(X, \mathcal{M}_X) \to (Spec\, k, \mathcal{M}_S)$ gives X a pre-stable structure, but it gives also a set of "special" points, namely $s_1, \ldots, s_n$ in the previous notation; such points should be thought as marked points for the curve $\underline{X}$. Indeed, if every fiber of $\underline{X} \to \underline{S}$ is a genus g curve, the s_i make possible to define sections $\Sigma_i : \underline{S} \to \underline{X}$, and hence recover a genus g, n-pointed structure on the curve X (see [Kat1], Proposition 1.7 for detailed discussion and proofs). We will refer to the points s_i as the *marked points* of the log curve.

We have seen how markings can be recovered from the log structure on a log curve. Next step is to define the stability condition in terms of Log Geometry. For this we need the following definition

Definition 6.2.4 (Logarithmic differentials)**.** *Given a morphism of log schemes $f : X \to Y$, there exists a $\mathcal{O}_X$-module $\Omega^1_{X/Y}$, called the sheaf of* logarithmic differentials*, which carries a universal derivation $(d, dlog)$ in the logarithmic category and verifies,* *for every $\mathcal{O}_X$-module A*

$$\mathrm{Hom}_{\mathcal{O}_X}(\Omega^1_{X/Y}, A) \simeq Der_Y(X, A)$$

as $\mathcal{O}_X$-modules.

In the case $Y = Spec\,(\mathbb{N} \to k)$ is the standard log point, X a smooth variety and $\mathcal{M}_X$ coming from a normal crossing divisor D, then the module $\Omega^1_{X/Y}$ is generated by the differentials with logarithmic poles along D (with a relation). For details and properties of $\Omega^1_{X/Y}$ see [Ogu].

Proposition 6.2.5 ([Kat1] 1.13)**.** *Given a log curve*

$$f : (X, \mathcal{M}_X) \to (S = Spec\,k, \mathcal{M}_S)$$

with marked points $\{s_1, \ldots, s_n\}$, there exists a natural isomorphism

$$\Omega^1_{X/S} \simeq \omega_{\underline{X}}(s_1 + \cdots + s_n).$$

From the previous Proposition one can see that the stability condition can be rephrased in terms of the sheaf of log differentials which naturally encodes the contribution of the markings. Hence we can give the following definition:

Definition 6.2.6 (Log stable curve)**.** *Given a log-curve $f : X \to S$ let $p \in S$ denote a close point; define $l = l(p)$ and $n = n(p)$ to be the integers such that*

$$\overline{\mathcal{M}}_{X_p/p} = \mathbb{Z}_{r_1} \oplus \cdots \oplus \mathbb{Z}_{r_{l(p)}} \oplus \mathbb{N}_{s_1} \oplus \cdots \oplus \mathbb{N}_{s_{n(p)}}.$$

Then f (X) is called a g, n log stable curve *if $\underline{f}$ is proper, $\underline{X}$ has genus g, $n(p) = n$ for every p and $\Omega^1_{X/S}$ is ample.*

6.2.2 Canonical log structure on stable curves

From the previous discussion it follows that every g, n log stable curve is (naturally) a genus g, n-pointed stable curve in usual sense. Next step would be a natural converse of the previous construction, i.e. given a usual stable curve, define a natural/canonical log structure so that the relative characteristic is supported at the nodes. Since we are going to use this construction in the sequel we briefly recall it.

Let $X \to S$ be a stable genus g curve with n markings. We are going to restrict to the special case $S = Spec\, k$ for a field k although the same construction can be done over the spectrum of a strict Henselian ring. Near a node r_i of f one can find a Cartesian diagram, over an étale neighborhood of (points specializing to) r_i, that looks as follows:

$$\begin{tikzcd}
U_i \arrow[r, "\psi_i"] \arrow[d] & Spec\, \dfrac{k[x,y,t]}{(xy-t)} \arrow[d, "\pi"] \\
S \arrow[r, "\varphi_i"] & Spec\, k[t]
\end{tikzcd}$$

Define on the bottom right corner a log structure $\mathbb{N} \to k[t]$ by $e \mapsto t$; in the same way on the upper right corner define a log structure

$$\mathbb{N}^2 \to \frac{k[x,y,t]}{(xy-t)}$$

by

$$e_1 \mapsto x \qquad\qquad e_2 \mapsto y$$

(e denotes generator for $\mathbb{N}$). Then if Δ is the diagonal $\mathbb{N} \to \mathbb{N}^2$, (π, Δ) is a log morphism. Using these structures we can pull them back to S and U_i via φ_i and ψ_i respectively, obtaining log structures $\mathcal{L}_i$ and $\mathcal{M}'_i$.

Away from r_i we define $\mathcal{M}''_i$ to be the pullback of $\mathcal{L}_i$; the log structure $\mathcal{M}_i$ on X will be defined as the gluing of $\mathcal{M}'_i$ and $\mathcal{M}''_i$. Finally let $\mathcal{N}$ be the log structure induced by the divisor of marked points.

Definition 6.2.7 (Canonical log structure on stable curves). *Given a stable curve $f : X \to S$ of genus g with n marked points,* *in the previous notations, the* canonical log structure *on f is the following couple of log structures on X and S respectively*

$$\mathcal{M}_X = \mathcal{M}_1 \oplus_{\mathcal{O}_X^*} \cdots \oplus_{\mathcal{O}_X^*} \mathcal{M}_l \oplus_{\mathcal{O}_X^*} \mathcal{N}$$
$$\mathcal{M}_S = \mathcal{L}_1 \oplus_{\mathcal{O}_X^*} \cdots \oplus_{\mathcal{O}_X^*} \mathcal{L}_l.$$

With this structure $f : (X, \mathcal{M}_X) \to (S, \mathcal{M}_S)$ is a log curve as in Definition 6.2.1 where the morphism at the level of log structure is naturally defined by construction (and by the Cartesian diagram above).

Remark 6.2.8. The canonical structure on the base $\mathcal{M}_S$ is locally free, i.e. there exists a non-negative integer l such that $\overline{\mathcal{M}}_{S,s} \simeq \mathbb{N}^l$ for every closed point $s \in S$. This gives a global chart $\mathbb{N}^l \to \mathcal{M}_S$. Now the analogous of the local picture described above in the neighborhood of a node, at the level of (pre) log structures looks as follows:

$$\begin{tikzcd}
\mathcal{M}|_U & \mathbb{N}^{l-1} \oplus \mathbb{N}^2 \arrow[l] \\
\pi^*(\mathcal{M}_S)|_U \arrow[u, "\pi^\flat"] & \mathbb{N}^{l-1} \oplus \mathbb{N} \arrow[l] \arrow[u, "(id,\Delta)"']
\end{tikzcd}$$

The image in $\overline{\mathcal{M}}_S$ of the generator of the bottom right $\mathbb{N}$ is called the element in $\overline{\mathcal{M}}_S$ *smoothing* the node p.

The canonical log structure verifies a *minimality* condition in the following sense: given a log curve $f : X \to S$, for any other log curve $f' : X' \to S'$ whose underline curve is isomorphic to the base change $X' \simeq X \times_S S'$ and such that the divisor of marked points of X' is sent scheme-theoretically to the divisor of marked points of X, there exist unique log maps α, β, extending the maps given by the fiber product, making the following diagram Cartesian

$$\begin{tikzcd}
X' \arrow[r, "\beta"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
S' \arrow[r, "\alpha"] & S
\end{tikzcd}$$

We end this section by explicitly determining the stalks of the characteristic at smooth points, marked points and nodes. This will give, in particular, a local description of the canonical structure over marked points and nodes in terms of the logarithmic structure of the base.

Remark 6.2.9. Given a log curve $(\mathcal{C}, \mathcal{M})$ with canonical log structure over a point $(Spec\, k, \mathcal{N})$, where $\mathcal{N}$ comes from a monoid P with the unit as the only invertible element, the characteristic of $\mathcal{M}$ at the level of stalks can be described as follows:

1. Over a generic point ξ, or a closed point different from nodes and marked points, the log structure comes entirely from below hence
$$\overline{\mathcal{M}}_{\xi} = P.$$

2. Over a marked point s the log structure comes from the base plus a contribution of the marked point. In particular, given Theorem 6.2.3 and Definition 6.2.7 one gets
$$\overline{\mathcal{M}}_{s} = P \oplus \mathbb{N}.$$

3. Finally over a node r the log structure has again a component coming from the base and a component depending on the node. From the description given at the beginning of this subsection of the local behavior of the canonical structure in a neighborhood of a node, one sees that
$$\overline{\mathcal{M}}_{r} = P \oplus_{\mathbb{N}} \mathbb{N}^2,$$
where the map $\mathbb{N} \to P$, given by $1 \mapsto p$, comes from $\mathcal{C} \to Spec\, k$ and the map $\mathbb{N} \to \mathbb{N}^2$ is the diagonal.

6.3 Log stable maps

In this section we are going to review the extension of the theory of stable maps to log stable maps via Logarithmic Geometry tools. Originated by the work of Kontsevich [Kon] to rigorously formulate (and check) prediction in enumerative geometry, the moduli space of stable maps, compactifying the space of maps from stable curves to varieties with fixed numerical data, gives the proper framework to build Gromov-Witten (GW) invariants' theory and their computation. Since the works [Li1] and [Li2] (in the algebraic category), relative GW invariants with respect to a smooth divisor has been extensively studied from different points of view. In a 2001 lecture Bernd Siebert proposed a new approach based on Logarithmic Geometry that, lately, led to establish the right settings for a good moduli theory of log stable maps [GS]. A different approach was introduced by Chen in [Che1] which uses marked graphs instead of insights from tropical geometry with some restriction on the target spaces. Both these approaches define a suitable condition on the log maps (*basicness* in Gross and Siebert's work, *minimality* in Chen's paper) which allows to define moduli spaces for these maps (with some extra conditions) which are proper and carry a perfect obstruction theory, as well as a virtual fundamental class.

For the purpose of the present work, we are mainly interested in Chen's approach, or better in its generalization [AC], since it deals precisely with the situation we are interested in, i.e. a target space where the log structure is given by a simple normal crossing divisor. We refer to the papers [Che1] and [AC] for detailed proofs.

6.3.1 Basic properties of log stable maps

We start by recalling that a *stable map* $f : \mathcal{C} \to Y$ is a diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "f"] \arrow[d, "\pi"'] & Y \\ S & \end{tikzcd}$$

with Y a projective scheme, $\pi : \mathcal{C} \to S$ a pre-stable curve and $f : \mathcal{C} \to Y$ a morphism; the stability condition requires that $\mathrm{Aut}_Y(\mathcal{C}_\xi)$ is finite for every geometric point $\xi \in S$. When considering moduli problems for stable maps one fixes the genus g and the number n of marked points of $\mathcal{C} \to S$ together with a curve class β in the projective scheme Y. Once these numerical data are assigned, there exists a Deligne-Mumford stack $\mathscr{M}_{g,n}(Y, \beta)$ parametrizing stable maps $f : \mathcal{C} \to Y$ from prestable curves $\mathcal{C}$ of genus g with n marked points such that $f_*[\mathcal{C}] = \beta$. We note briefly that pre-stable maps form a category fibered in grupoids over **<u>Sch</u>**, the category of schemes, and that this category is represented by an algebraic stack which contains, as a *proper* substack, stable maps of fixed numerical data (g, n, β).

We begin by an extension of this definition to the logarithmic category.

Definition 6.3.1. *A* pre-stable log map $f : \mathcal{C} \to Y$ *over a fs log scheme S is a couple given by a log curve $\pi : \mathcal{C} \to S$ and a morphism of log schemes $\mathcal{C} \to Y$. It can be represented by the following diagram:*

$$\begin{tikzcd}
(\mathcal{C}, \mathcal{M}_{\mathcal{C}}) \arrow[r, "f"] \arrow[d, "\pi"'] & (Y, \mathcal{M}_Y) \\
(S, \mathcal{M}_S) &
\end{tikzcd}$$

where now the diagram lives in the logarithmic category. A prestable log map is called stable *if the underlying map between schemes is stable in the usual sense.*

Morphisms of (pre-)stable curves are morphisms of log curves as in Definition 6.2.2 that are compatible with the log map; more precisely given two pre-stable log maps $f_1 : (\mathcal{C}_1, \mathcal{M}_1) \to (Y, \mathcal{M}_Y)$ and $f_2 : (\mathcal{C}_2, \mathcal{M}_2) \to (Y, \mathcal{M}_Y)$ over $(S, \mathcal{M}_{S_1})$ and $(S, \mathcal{M}_{S_2})$ respectively, an isomorphism of pre-stable log maps is an isomorphism of the log-

curves (σ, γ) such that the following diagram is commutative:

$$\begin{tikzcd}
 & & (Y, \mathcal{M}_Y) \\
(\mathcal{C}_1, \mathcal{M}_1) \arrow[r, "\gamma"] \arrow[d] \arrow[urr, bend left, "f_1"] & (\mathcal{C}_2, \mathcal{M}_2) \arrow[d] \arrow[ur, bend right, "f_2"'] & \\
(S, \mathcal{M}_{S_1}) \arrow[r, "\sigma"] & (S, \mathcal{M}_{S_2}) &
\end{tikzcd}$$

Previous Definition can be extended to deal with the case of families of targets, i.e. replacing the fs log scheme Y with a family $Y \to B$ of projective log schemes (see [Che1] section 2 and [AC] section 5). Objects and arrows of the corresponding category are defined similarly to the previous Definition.

In the same spirit of Remark 6.2.9 we give a local description of a log map in the case where the log structure $\mathcal{M}_Y$ on the target Y is given by a smooth divisor D. This gives a map $\mathbb{N} \to \overline{\mathcal{M}}_Y$ (see example 6.1.10). This log structure (and more generally every locally free DF-structure (see next section)) is equivalent to the data of a line bundle $\mathcal{L}$ (in this case $\mathcal{O}_Y(-D)$) and a section $s \in H^0(\mathcal{L}^\vee)$ which gives a map $\mathcal{L} \to \mathcal{O}_Y$ that induces the logarithmic structure on Y.

Remark 6.3.2. Let $f : \mathcal{C} \to Y$ be a log map over a geometric point $S = Spec\, k$ with log structure $\mathcal{M}_S$. Suppose that the log structure $\mathcal{M}_Y$ on Y is given by a smooth divisor $D \subset Y$ as above and that the log structure of the curve $\mathcal{C} \to \mathcal{S}$ is the canonical one. For every point $p \in \mathcal{C}$ in an irreducible component $Z = Z_p \subset \mathcal{C}$ there is a map at the level of the characteristics

$$\overline{f}^\flat : f^*(\overline{\mathcal{M}}_Y)_p \to \overline{\mathcal{M}}_{C,p}.$$

As in remark 6.2.9 we focus on three cases that exhaust all the possible behaviors of the map $\overline{f}^\flat$.

1. Let p be a smooth, non-marked point. Then the map

reads
$$\overline{f}^{\flat} : f^*(\overline{\mathcal{M}}_Y)_p \to \overline{\mathcal{M}}_S.$$
If δ denotes the standard generator of $\mathbb{N}$ then we have that $\overline{f}^{\flat}(\delta) = e \in \overline{\mathcal{M}}_S$. Such e is called the *degeneracy* of the irreducible component $Z = Z_p$. By the description of $\mathcal{M}_Y$, if $f(p) \notin D$ for all $p \in Z$ then the image of $\overline{f}^{\flat}$ vanishes. We call a component Z *degenerate* if its degeneracy is not zero.

2. Let p be a marked point. Now the maps reads
$$\overline{f}^{\flat} : f^*(\overline{\mathcal{M}}_Y)_p \to \overline{\mathcal{M}}_S \oplus \mathbb{N}$$
and then, with the above notation we have that
$$\overline{f}^{\flat}(\delta) = e + c_p \cdot \sigma_p,$$
where c_p is a non negative integer and σ_p denotes the generator of $\mathbb{N}$ in the stalk of the characteristic of $\mathcal{C}$. The integer c_p is called the *contact order* of f at p. It can be identify with the multiplicity of intersection $(f(\mathcal{C}) \cdot D)_{f(p)}$.

3. Let p be a node, at the intersection of two irreducible components Z and Z'. Let e_p be a smoothing of the node p, and let $\log x_p$ and $\log y_p$ be elements in $\overline{\mathcal{M}}_{\mathcal{C}}$ given by the local coordinates of Z and Z' respectively, at p. Then by the construction of the canonical log structure of $\mathcal{C} \to \mathcal{S}$ the following equality holds in $\overline{\mathcal{M}}_{\mathcal{C}}$:
$$e_p = \log x_p + \log y_p.$$
Hence, without loss of generality, the map $\overline{f}^{\flat}$ can be assumed to be
$$\overline{f}^{\flat}(\delta) = e + c_p \cdot \log x_p$$
for a non negative integer c_p, called the *contact order* of f at the node p. The node is called *distinguished* if $c_p \neq 0$.

The degeneracy of the two components Z and Z' are respectively e and $e + c_p \cdot e_p$. Sometimes Z is called the *lower* component and Z' the *upper* component of p

The main property of the contact orders is summarized in the following

Lemma 6.3.3. *Give a log map $f : \mathcal{C} \to Y$ over S the following hold:*

1. *for every marking Σ_i, there exists an open subset S' such that the order of contact along Σ_i is constant;*
2. *for every node at the intersection of two irreducible component there exists an open subset S'' such that either the node is smoothed out or its contact order remains the same.*

6.3.2 Deligne Falting pairs and moduli of minimal log stable maps

As for usual stable maps one can construct the moduli space associated to log (stable) maps, which is a category fibered in grupoids over a log scheme S with fixed genus and marked points. This stack, denoted by $\mathcal{LM}_{g,n}(Y)$ (or $\mathcal{LM}_{g,n}(Y/B)$ in the relative setting), is a category fibered in grupoids that associates to every log scheme (or B-scheme) S the grupoids of log maps over S, such that the underlying map has source given by a pre-stable genus g curve with n marked points. This stack is algebraic in the sense of Artin ([Che1]) but is, nonetheless, unhandy, containing all possible log structure on the base S. Since the main goal is to study log maps without the information on the base structure we will focus on two open substacks parametrizing a special subclass of log maps called minimal.

A complete treatment of minimality for log stable maps is beyond the scope of this chapter, nevertheless we gave here the basic definition and results referring to [Che1] section 3 and [AC] for detailed construction and proofs.

Before moving to minimality we fix the settings in which such a notion is naturally defined. We begin by the following

Definition 6.3.4. *Given a log scheme $(X, \mathcal{M}_X)$ we call it a* Deligne-Faltings pair*, often written as DF pair, if $\mathcal{M}_X$ is locally free and there is a morphism of locally constant sheaves $\mathbb{N}^k \to \overline{\mathcal{M}}_X$ which locally lifts to a chart. The map $\mathbb{N}^k \to \overline{\mathcal{M}}_X$ is called a* global presentation *of $\mathcal{M}_X$ and the integer k is called the* rank *of the Deligne-Falting pair $(X, \mathcal{M}_X)$.*

As a typical example take a smooth Cartier divisor D in a smooth scheme X, and equip X with the log structure $\mathcal{M}_X$ coming from D. Then $(X, \mathcal{M}_X)$ is a Deligne-Faltings pair of rank 1, and $\mathcal{M}_X$ corresponds to the line bundle $\mathcal{O}_X(-D)$ with the map $\mathcal{O}_X(-D) \to \mathcal{O}_X$.

For the rest of this chapter assume fixed a rank k DF pair $(Y, \mathcal{M}_Y)$ with $\underline{Y}$ projective and global presentation of $\mathcal{M}_Y$ given by $\mathbb{N}^k \to \overline{\mathcal{M}}_Y$. In the case $k = 1$ to such a pair it corresponds the data of a line bundle $\mathcal{L}$ and a section $s \in H^0(\mathcal{L}^\vee)$ which give a map of sheaves $\mathcal{L} \to \mathcal{O}_Y$ whose associated log structure is $\mathcal{M}_Y$. If s is not a zero section, then, denoted by D the vanishing locus of s, $\mathcal{L} = \mathcal{O}_Y(-D)$. Moreover if δ is the generator of $\mathbb{N}$, which we will identify by abuse of notation with its image in $\overline{\mathcal{M}}_Y$, we have that δ lifts, locally, to a section of $\mathcal{O}_Y$ whose vanishing locus gives the divisor D.

If $k > 1$ then for $i = 1, \ldots, k$ one has maps $\alpha_i : \mathbb{N} \to \mathbb{N}^k$ given by the inclusion of the i-th component. The composition

$$\mathbb{N} \xrightarrow{\alpha_i} \mathbb{N}^k \longrightarrow \mathcal{M}_Y$$

defines a rank 1 DF structure $\mathcal{M}_{Y,i}$ on $\underline{Y}$ which gives a decomposition

$$\mathcal{M}_Y = \bigoplus_{i=1}^{k} \mathcal{M}_{Y,i}$$

in which each $\mathcal{M}_{Y,i}$, by the discussion above, corresponds to a couple $(\mathcal{L}_i, s_i)$ of line bundles and sections. As an example of such a construction, generalizing the above example, one can consider a simple normal crossing divisor in which each $\mathcal{L}_i$ corresponds to $\mathcal{O}_X(-D_i)$, for an irreducible component D_i of D.

The first definition of minimality follows from the log structure of the moduli space in the following way: consider the category $\mathcal{LM}_{g,n}(Y)$ fibered over fs log scheme, instead of log schemes. Such a category admits a fs logarithmic structure as an Artin stack; denote it by $\mathcal{K}(Y) = (\mathcal{K}, \mathcal{M}_{\mathcal{K}})$: stable log maps over S are equivalent to log morphism $S \to \mathcal{K}(Y)$. The existence of such a fs log stack permits to identify a *minimal* structure on log maps over usual schemes by pulling back the log structure $\mathcal{M}_{\mathcal{K}}$.

More formally, given a scheme $\underline{S}$, an object of the stack $\underline{\mathcal{K}}(Y)$ is an arrow $\underline{S} \to \underline{\mathcal{K}}(Y)$. This map gives a log scheme $S = S^{\min}$ where the log structure is pulled back from $\mathcal{K}(Y)$. Now from every log map over S, $f : \mathcal{C} \to Y$, one obtains a log map with log structure pulled back from $\mathcal{K}(Y)$ and its universal family $\mathfrak{C}$. This can be made explicit in the following Cartesian diagram:

$$\begin{tikzcd}
\mathcal{C} \arrow[r] \arrow[d] & \mathcal{C}^{\min} \arrow[r] \arrow[d] & \mathfrak{C} \arrow[r] \arrow[d] & Y \\
S \arrow[r] \arrow[dr] & S^{\min} \arrow[r] \arrow[d] & \mathcal{K}(Y) \arrow[d] & \\
 & \underline{S} \arrow[r] & \underline{\mathcal{K}}(Y) &
\end{tikzcd}$$

Definition 6.3.5. *A log map $f : \mathcal{C} \to Y$ over S is called* minimal *if the log structure on $\mathcal{C} \to S$ is pulled back from the log structure of $\mathcal{K}(Y)$. This is equivalent to require that the map $S \to \mathcal{K}(Y)$ corresponding to f is strict.*

Minimality can be checked combinatorially from the map $f : \mathcal{C} \to Y$ without any reference to the log structure of the stack $\mathcal{K}(Y)$. This has the advantage that, given any log map, one can define explicitly a minimal log structure associated to f. In fact one can do better and obtaining the following:

Theorem 6.3.6 (Q. Chen). *To any log map $f : (\mathcal{C}, \mathcal{M}_C) \to Y$ over an fs log scheme $(S, \mathcal{M}_S)$ there exists a minimal log map over $(S, {\mathcal{M}_S}^{\min})$, and a map of fs log schemes*

$$\Phi : (S, \mathcal{M}_S) \to (S, {\mathcal{M}_S}^{\min}),$$

both unique up to isomorphism, such that the following diagram is commutative

$$\begin{tikzcd}
 & & (Y, \mathcal{M}_Y) \\
(\mathcal{C}, \mathcal{M}_{\mathcal{C}}) \arrow[r, "\Phi_{\mathcal{C}}"] \arrow[d] \arrow[urr, bend left, "f"] & (\mathcal{C}, {\mathcal{M}_{\mathcal{C}}}^{\min}) \arrow[d] \arrow[ur, bend right, "f_{\min}"'] & \\
(S, \mathcal{M}_S) \arrow[r, "\Phi"] & (S, {\mathcal{M}_S}^{\min}) &
\end{tikzcd}$$

Construction 6.3.7. We briefly sketch the ideas underlying Chen's combinatorial description of minimality for the case $k = 1$. To each log map $f : \mathcal{C} \to Y$ over S there is a canonical graph associated, namely the dual graph of the curve $\mathcal{C}$. To such a graph, Chen gives weights to the edges given by contact orders and an orientation to a subset of them coming from the partial order defined by upper and lower components; he then constructs a monoid $\mathcal{M}(G_f)$ which posses a generator for each vertex and edge together with some natural relations coming from the local behavior of the map at the level of characteristic described in subsection 6.3.1. By quotienting out the torsion part of the associated group and taking the saturation of the image of the monoid $\mathcal{M}(G_f)$ in such a quotient one obtains a sharp monoid, $\overline{\mathcal{M}}(G_f)$ with a canonical map

$$\phi : \overline{\mathcal{M}}(G_f) \to \overline{\mathcal{M}}_S.$$

Then the map f is minimal if and only if ϕ is an isomorphism.

We now fix discrete data that gives a partition of the moduli space of log stable maps.

Notation 6.3.8. Denote by Γ the fourple $(\beta, g, n, \vec{c})$ where

1. $\beta \in H^2(\underline{Y}, \mathbb{Z})$ is a curve class;
2. g is a non-negative integer denoting the genus;
3. n is a non-negative integer denoting the number of marked points;
4. $\vec{c}$ is a set of tuples, denoting the contact order of the markings. In particular $\vec{c} = (c_{ij})$ with $i = 1, \dots, k$ and $j = 1, \dots, n$ such that

$$\beta \cdot \mathbf{c_1}(L_i^\vee) = \sum_{j=1}^{n} c_{ij}.$$

Here the L_i are the line bundles associated to the DF pair $(Y, \mathcal{M}_Y)$ of rank k, and $\mathbf{c_1}$ denote the first Chern class.

Definition 6.3.9. *A minimal log stable map $f : \mathcal{C} \to Y$ over a geometric point S is called minimal* with respect to Γ, *or* Γ-*minimal if*

1. *the curve $\mathcal{C} \to S$ is a log pre-stable curve of genus g with n-markings;*
2. *the pushforward of the fundamental class of $\mathcal{C}$ equals β, i.e. $f_*[\mathcal{C}] = \beta$;*
3. *the contact order along the i-th marking with the j-th component D_j of the divisor D is given by c_{ij}.*

The stack parametrizing minimal stable maps with genus g, n-markings and curve class β is denote by $\mathcal{K}_{g,n}(Y, \beta)$ and the stack parametrizing Γ*-minimal stable maps is denoted by $\mathcal{K}_\Gamma(Y)$.*

The above stacks are both algebraic and the following partition, of open and close substacks holds:

$$\mathcal{K}_{g,n}(Y, \beta) = \bigsqcup_{\Gamma} \mathcal{K}_\Gamma(Y).$$

The main properties of the above stacks are summarized in the following

Theorem 6.3.10 (Q. Chen, Abramovich - Chen). *The fibered category $\mathcal{K}_\Gamma(Y)$ is a proper Deligne-Mumford stack with a representable and finite map to the stack of standard stable maps obtained by removing log structures.*

Remark 6.3.11. The projectivity of the scheme $\underline{Y}$ is essential for the properness of the stack; however some properties can be recovered with the weaker hypothesis of Y being separated and of finite type over $\mathbb{C}$.

7

Complements of a very generic quartic

The main goal of this chapter is to extend Theorem 3.2.3 to complements of very generic quartics. The main idea is to consider flat deformations of degree four divisors in $\mathbb{P}^2$ and reformulate Corvaja and Zannier result in order to make it invariant under such deformations of the boundary divisor. The tool used here is Logarithmic Geometry in the sense of Kato and Illusie as introduced in Chapter 6. We will begin by an extension of the result for complements of a conic and two lines to log-stable maps. The aim is to give a "moduli" reformulation relating the degree bound to a vanishing of certain moduli space of log stable maps to DF pair as defined by Chen and Abramovich-Chen (see Section 6.3). This will be the focus of the second section where we will prove that the strengthen version of Theorem 3.2.3 is actually equivalent to the emptiness of some stacks of (minimal) log stable maps. Once this equivalence is settled we use the properness of the stacks to obtain the same vanishing for stacks constructed from complements of more general degree four divisor.

7.1 Three components case for log-stable maps

Corvaja and Zannier prove Lang-Vojta Conjecture in the split function field case for the complement of a conic and two lines in $\mathbb{P}^2$ as described in detail in section 3.2.1 at page 50. Their main result, i.e. Theorem 3.2.3 page 50 states that morphisms $f : \tilde{\mathcal{C}} \to \mathbb{P}^2$ such that $\tilde{\mathcal{C}}$ is a smooth projective curve, $f^{-1}(D) \subset S$ for a divisor D in $\mathbb{P}^2$ consisting of a conic and two lines with normal crossing and a finite set of points $S \subset \tilde{\mathcal{C}}$, verify

$$\deg(f(\tilde{\mathcal{C}})) \leq 2^{15} \cdot 35 \cdot \max\{1, \chi(\tilde{\mathcal{C}} \setminus S)\},$$

where as usual the *Euler characteristic* of the affine curve $\tilde{\mathcal{C}} \setminus S$ is defined as

$$\chi(\tilde{\mathcal{C}} \setminus S) := \chi_S(\tilde{\mathcal{C}}) = 2g(\tilde{\mathcal{C}}) - 2 + \#S.$$

The first step towards a generalization of Theorem 3.2.3 is to consider more general maps than the one considered by Corvaja and Zannier. The reason is that we want to use deformation arguments applied to our situation and this led naturally to consider reducible curves.

7.1.1 Extending Corvaja and Zannier Theorem

Consider a flat deformation D_t of the divisor D, a conic and two lines in general position, in which the general member is a quartic and the special fiber over a geometric point is $D_0 = D$. Construct the associated fibered threefold $\mathcal{X}$ in which every fiber is $\mathbb{P}^2$ with the corresponding divisor D_t. We want to study families of curves that moves in the deformation, i.e. such that when restricted to a fiber give a plane curve. In particular we want to answer the following questions:

Q1: Suppose that the image in the general fiber is smooth: what can be sad about the image in the special fiber $\mathbb{P}^2, D_0$?

A: From the theory of stable map we know that such a curve could acquire at most nodal singularities and in particular can become reducible.

Q2: What can be said about the variation of the number of points of intersection with the divisor D_t?

A: The question here is more subtle: the main difficulty is that multiplicities of intersection cannot be controlled *a priori* in the family. Even worse one of the components of the curve can be mapped *on* a component of the boundary divisor. In such a case even defining the multiplicity of intersection can be troublesome. That is precisely the point in which Logarithmic Geometry will play an important role.

From the previous discussion there is some evidence that we have to deal with morphisms from pre-stable curves, i.e. reducible connected curves with at worst nodes as singularities. The first thing we have to care about is the behavior of the set S, i.e. the set of points in the curves that are mapped to the divisor, when considering reducible curves instead of irreducible ones. With this goal in mind we first define the concept of partial normalization of a curve.

Definition 7.1.1. *Let C be an affine, reducible and at worst nodal curve. We resolve the nodal singularities which do not occur as points of intersection of different components. The strict transform of C gives rise to a curve of the form $\tilde{C} \setminus S$ for a projective, reducible and at worst nodal curve $\tilde{C}$ in which every node occurs as a point of intersection of two different components; we call $\tilde{C} \setminus S$ a* partial normalization *of C. In particular all partial normalized curves have simple normal crossing singularities.*

As an example consider the figure 7.1 in which a blow-up of a curve $\tilde{C}$ at the node P is pictured: the blow-up resolves singularities in the irreducible components distinct from the normal crossing singularities arising as intersection between different components.

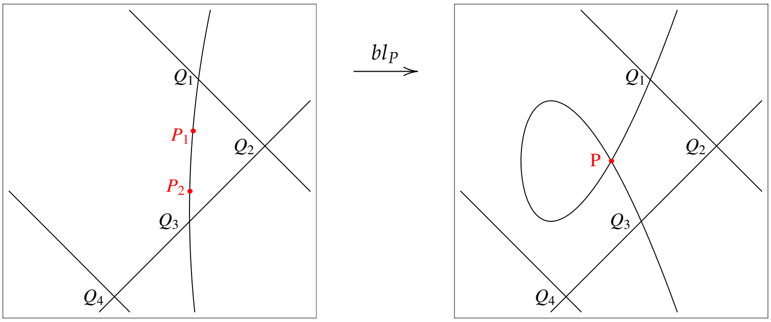


Figure 7.1: Partial normalization of a curve

Now from a partial normalization of $\mathcal{C}$ we would like to consider the set S such that it is additive with respect to the irreducible components of $\mathcal{C}$, i.e. if $\mathcal{C}_1, \ldots, \mathcal{C}_r$ are the irreducible components of the partial normalization of the pre-stable curve $\mathcal{C}$ and S_i denotes the set of points of S that lie in the irreducible component $\mathcal{C}_i$, we want to understand whether is it true that

$$\sum_{i=1}^{r} \#S_i \stackrel{?}{=} \#S.$$

It is easy to construct examples in which the former equality does not hold. Consider a two component plane curve $\mathcal{C}_1 \cup \mathcal{C}_2$ in which each component is isomorphic to a line and they intersect in a unique node $P = \mathcal{C}_1 \cap \mathcal{C}_2$. Suppose that the only point of S in the curve is the node. Then one has

$$\#S_1 = 1 \qquad \#S_2 = 1 \qquad \#S = 1.$$

As this example shows, the problems can arise when considering curves in which the points of S are nodes. In facts nodes are special points of two different irreducible components and therefore they count twice when considered in each component separately, but at the same time, they are counted once in the total curve. More in general one can prove that if the set S does not contain any node then its cardinality is additive in the previous sense. On the other hand, when the set S contains nodes, the additivity fails to hold. Nevertheless one can relate the total number of points of the set S with the sum of the number of points in each irreducible component in the following way:

Lemma 7.1.2. *Let $\mathcal{C}$ be the partial normalization of a pre-stable curve and let S as before. Let $\mathcal{C}_i$ be the i-th irreducible component of $\mathcal{C}$ for $i = 1, \ldots, r$ and let $S_i = S \cap \mathcal{C}_i$, i.e. the set of points of S in the irreducible component $\mathcal{C}_i$, then the following holds:*

$$\sum_{i+1}^{r} \#S_i \leq \#S + \#\ \textit{nodes}\ .$$

Proof. For previous discussion, if S does not contain any node then $\#S$ is additive in its components and hence the conclusion holds trivially. Suppose that S contains exactly a node P: then such a node is a point of at most two components, say $\mathcal{C}_i$ and $\mathcal{C}_j$. This implies that

$$S_i \cup S_j = (S_i \setminus \{P\}) \cup (S_j \setminus \{P\}) \cup \{P\}$$

and then, since

$$\#S_i + \#S_j = \#S \cap (\mathcal{C}_i \cup \mathcal{C}_j) + 1,$$

one gets

$$\sum_{k=1}^{r} \#S_k = \#S + 1.$$

From the previous calculation we get that if S contains n nodes then sum of the number of points of S in each component is precisely equal to $\#S + n$. Hence, in the worst situation in which S contains all the nodes we get the upper bound required. □

Remark 7.1.3. From the proof of Lemma 7.1.2 we actually proved something more: we showed that if n is the number of nodes contained in S then

$$\#S = \sum \#S_i - n,$$

which in particular implies that the cardinality of S is additive if no nodes are contained in S. However, for future needs, we will need only the estimate given in the previous Lemma.

Next step is to study how the Euler characteristic behave for reducible curves with respect to the Euler Characteristic of the irreducible components. We split the description in the following two lemmas, treating first the case in which all the irreducible components have positive Euler characteristic.

Lemma 7.1.4. *Let $\tilde{C} = \cup_{i=1}^n C_i$ ($n \geq 2$) be a projective, reducible and at worst nodal curve, in which every node occurs as a point of intersection of two different components; let $g = g(\tilde{C})$, $g_i = g(C_i)$ and let S be a finite set of points on $\tilde{C}$ with $S_i = S \cap C_i$. If $\chi_{S_i}(C_i) > 0$ for all i, then the following holds:*

$$\sum_{i=1}^n \max\{1, 2g_i - 2 + \#S_i\} \leq 2g - 2 + \#S \qquad \textit{i.e.}$$

$$\sum_{i=1}^n \max\{1, \chi_{S_i}(C_i)\} \leq \chi_s(\tilde{C}).$$

In particular $\chi_S(\tilde{C}) \geq 1$.

Proof. The starting point is the formula for the genus of a connected at worst nodal projective curve which reads as follows:

$$g(\tilde{C}) = \sum_{i=1}^n g(C_i) + \#nodes - \#components + 1. \tag{7.1}$$

From this and from Lemma 7.1.2 we get that

$$\begin{aligned}
\chi_S(\tilde{C}) &= 2g - 2 + \#S \\
&= 2\sum_{i=1}^n g_i + 2\#\text{ nodes } - 2\#\text{ comp } + \#S \\
&\geq 2\sum_{i=1}^n g_i + 2\#\text{ nodes } - 2\#\text{ comp } + \sum \#S_i - \#\text{ nodes} \\
&\geq \sum_{i=1}^n \chi_{S_i}(C_i) + \#\text{ nodes }.
\end{aligned}$$

Now, we are assuming that for every component $\chi_{S_i}(\mathcal{C}_i) \geq 1$. Then

$$\begin{aligned}\chi_S(\tilde{\mathcal{C}}) &\geq \sum_{i=1}^{n} \chi_{S_i}(\mathcal{C}_i) + \#\,\text{nodes} \\ &\geq \sum_{i=1}^{n} \max\{1, \chi_{S_i}(\mathcal{C}_i)\}.\end{aligned}$$

□

Previous Lemma leaves out the case in which irreducible components of the curve have non positive Euler Characteristic. From geometric construction one can see that the only case that do not follow in the previous result is the one in which one of the components has genus 0 and only two points of S. In this case one cannot recover the same result but only a optimal estimate that read as follows:

Lemma 7.1.5. *With the notation above, without any restriction on the Euler characteristic of the irreducible components the following holds:*

$$\chi_S(\tilde{\mathcal{C}}) \geq \sum_{i=1}^{n} \chi_{S_i}(\mathcal{C}_i) + \#\,\textit{nodes}\ .$$

In particular, every time there exist more than one irreducible component, the Euler characteristic $\chi_S(\tilde{\mathcal{C}})$ is always positive.

Proof. The desired bound follows from Lemma 7.1.4 together with the lower bound:

$$\#\,\text{nodes} \ \geq \#\,\text{components}\ - 1.$$

□

7.1.2 Extension to log stable maps

As mentioned at the beginning of this section we want to extend Theorem 3.2.3 (page 50) to a more general class of maps. This class of maps will be the class of *logarithmic stable maps* introduced in Section 6.3 Chapter 7. This choice is natural in the following sense: maps considered by Corvaja and Zannier were morphism from abstract nonsingular curves $\tilde{\mathcal{C}}$ with a distinguished set of points S (whose complement corresponded to the normalization of an affine curve) to $\mathbb{P}^2$, such that the inverse image of the intersection between the (image of) the curve and the divisor D (a conic and two lines in general position) was precisely S. In this setting the target $(\mathbb{P}^2, D)$ of such maps is a Deligne-Falting pair, being D a simple normal crossing divisor. Moreover the divisor S on the source curve gives a log-structure over $\tilde{\mathcal{C}}$, and extends the map to a logarithmic map being S precisely the inverse image of D. In particular the map between log structures is strict and the log map is automatically stable, because we require the source to be irreducible, non singular and the map to be non-constant. For this reasons, having in mind a moduli construction, we want to study whether the conclusion of Corvaja and Zannier's Theorem holds more in general for a generic stable log-map $(\tilde{\mathcal{C}}, S) \to (\mathbb{P}^2, D)$.

We are then concerned with log-morphisms $\varphi : \tilde{\mathcal{C}} \to \mathbb{P}^2$ such that the pre-image of points of intersection $\varphi(\tilde{\mathcal{C}}) \cap D$ are contained in S. For our needs being a stable log-morphism from the curve $\tilde{\mathcal{C}}$ to $\mathbb{P}^2$ with respect to the divisor D is equivalent to the following conditions:

1. the underlying map between schemes is stable;
2. For every irreducible component $\mathcal{C}_i$ of $\tilde{\mathcal{C}}$ such that $\mathcal{C}_i$ maps to a degree one irreducible component of D, S_i contains at least **three** points.
3. For every irreducible component $\mathcal{C}_j$ of $\tilde{\mathcal{C}}$ such that $\mathcal{C}_j$ maps to the degree two irreducible component of D, S_j contains at least **four** points.

Here $S_i \subset S$ is again the set of points of S lying in the irreducible component $\mathcal{C}_i$, i.e. $S_i = S \cap \mathcal{C}_i$. Then extension of

Theorem 3.2.3 follows from the following theorem (from now on, in order to simplify the notation we put $A = 2^{15} \cdot 35$).

Theorem 7.1.6. *Given $\tilde{\mathcal{C}}, S, D$ as above, let $\varphi : \tilde{\mathcal{C}} \to \mathbb{P}^2$ be a non-constant log-morphism such that $\varphi^{-1}(D) \subset S$. Then the degree of the image $\varphi(\tilde{\mathcal{C}})$ verifies:*

$$\deg(\varphi(\tilde{\mathcal{C}})) \leq A \cdot \max\{1, \chi(\tilde{\mathcal{C}} \setminus S)\}.$$

Proof. If $\tilde{\mathcal{C}}$ is irreducible the conclusion is given by Theorem 3.2.3. Hence we will assume that $\tilde{\mathcal{C}}$ has more than one irreducible component. From Lemma 7.1.5 this implies that the Euler Characteristic $\chi_S(\tilde{\mathcal{C}})$ is strictly positive.

We divide the proof in three steps:

Step 0 We begin by proving the theorem for the case in which $\tilde{\mathcal{C}}$ has precisely two irreducible components $\tilde{\mathcal{C}} = \mathcal{C}_1 + \mathcal{C}_2$ such that $\varphi(\mathcal{C}_1) = D_1 = \{\text{line}\}$, and $\varphi(\mathcal{C}_2)$ meets properly the divisor D with $\chi_{S_2}(\mathcal{C}_2) > 0$. This will illustrate the main ideas of the proof in a handy case, before considering the general situation. Denote as usual S_1, S_2 as the set of points of S in the two irreducible components $\mathcal{C}_1$ and $\mathcal{C}_2$ respectively. In this case, being φ a log-morphism, we have $\#S_1 \geq 3$. Define $f_i := \varphi|_{\mathcal{C}_i}$ for $i = 1, 2$; in this setting we obtain

$$\begin{aligned} \deg \varphi(\tilde{\mathcal{C}}) &= \deg f_1(\mathcal{C}_1) + \deg f_2(\mathcal{C}_2) \\ &= 1 + \deg f_2(\mathcal{C}_2). \end{aligned}$$

We turn our attention to the component $\mathcal{C}_2$: the image of this smooth curve has proper intersection with the divisor D and hence falls into the cases of Theorem 3.2.3. Thus

$$\deg f_2(\mathcal{C}_2) \leq A \cdot \max\{1,\ 2g(\mathcal{C}_2) - 2 + \#S_2\}.$$

At the same time we have Equation 7.1 for the genus of a connected at worst nodal curve, from which, together with the stability condition, we get

$$\begin{aligned}\deg \varphi(\tilde{\mathcal{C}}) &= \deg f_1(\mathcal{C}_1) + \deg f_2(\mathcal{C}_2)\\ &\leq 1 + A\max\{1,\ 2g(\mathcal{C}_2) - 2 + \#S_2\}\\ &\leq A[2g(\mathcal{C}_1) - 2 + 3] + A\max\{1,\ 2g(\mathcal{C}_2) - 2 + \#S_2\}\\ &\leq A[2g(\mathcal{C}_1) - 2 + \#S_1] + A\max\{1,\ 2g(\mathcal{C}_2) - 2 + \#S_2\}\\ &\leq A[2g(\mathcal{C}_1) - 2 + \#S_1] + A[2g(\mathcal{C}_2) - 2 + \#S_2]\\ &\leq A\big(2[g(\mathcal{C}_1) + g(\mathcal{C}_2)] - 2\cdot 2 + \#S_1 + \#S_2\big).\end{aligned}$$

Using Lemma 7.1.2, which gives $\#S_1 + \#S_2$ bounded above by $\#S$ plus the number of nodes, and the formula for the genus of a pre-stable curve we conclude that

$$\begin{aligned}\deg \varphi(\tilde{\mathcal{C}}) &\leq A\big(2[g(\mathcal{C}_1) + g(\mathcal{C}_2)] - 2\#\text{ comp } + \#S + \#\text{ nodes }\big)\\ &\leq A\big(2[g(\mathcal{C}_1) + g(\mathcal{C}_2)] - 2\#\text{ comp } + 2\#\text{ nodes } + \#S\big)\\ &\leq A\big(2g(\tilde{\mathcal{C}}) - 2 + \#S\}\\ &\leq A\max\{1,\ 2g(\tilde{\mathcal{C}}) - 2 + \#S\}.\end{aligned}$$

We observe that, as in the proof of lemma (7.1.4) we can always assume $\#S_2 \geq 2$; moreover the same proof could be applied in the case of a two-component curve in which one of the two components maps over the degree-two component of the divisor D.

Step 1 Having Step 0 we apply the same ideas to a more general situation. We start from a connected at worst nodal affine curve whose partial normalization, in the sense defined above, is $\tilde{\mathcal{C}} \setminus S$ where $\tilde{\mathcal{C}}$ is a connected at worst nodal projective curve in which the nodes occur only as intersection of two different components. For a fixed log-morphism $\varphi : \tilde{\mathcal{C}} \to \mathbb{P}^2$ such that $\varphi^{-1}(D) \subset S$, we arrange the components of $\tilde{\mathcal{C}}$ in four groups so that

$$\tilde{\mathcal{C}} = \sqcup\, \mathcal{C}_i \sqcup \mathcal{G}_j \sqcup \mathcal{D}_h \sqcup \mathcal{E}_k,$$

where the image of the components $\mathcal{C}_i$ meets properly the divisor D and $\chi_{S_i}(\mathcal{C}_i) > 0$, the image of the components $\mathcal{G}_j$ meets properly the divisor D and $\chi_{S_j}(\mathcal{C}_i) = 0$ (i.e. curves of genus 0 with two points in each S_j), $\varphi(\mathcal{D}_h)$ overlaps a line of the divisor D and $\mathcal{E}_k$ is mapped over the degree-two component of D. We note that one can prove that in the three components case the only images of curves with zero Euler characteristic are lines; in particular their degree is always strictly less than 2. As before, for every component $\mathcal{C}_i$ we can apply Theorem (3.2.3) which gives

$$\deg \varphi(\mathcal{C}_i) \leq A \cdot \max\{1,\ 2g(\mathcal{C}_i) - 2 + \#S_i\}.$$

Where S_i is the set of points of S lying on $\mathcal{C}_i$. For the sets S and S_i we have the following estimate

$$\#S \geq \sum_{i:\mathcal{C}_i \in \tilde{\mathcal{C}}} \#S_i + \sum_{j:\mathcal{G}_j \in \tilde{\mathcal{C}}} 2 + \sum_{h:\mathcal{D}_h \in \tilde{\mathcal{C}}} 3 + \sum_{k:\mathcal{E}_k \in \tilde{\mathcal{C}}} 4 - \#\,\text{nodes}\,.$$

As before we want to relate the genus of the curve $\tilde{\mathcal{C}}$ to the genus of its irreducible components; Equation 7.1 in this setting gives

$$g(\mathcal{C}) = \sum_{\mathcal{A} \in \tilde{\mathcal{C}}} g(\mathcal{A}) + \#\,\text{nodes}\ - \#\,\text{components}\ + 1$$

where the sum runs over all the irreducible components of $\tilde{\mathcal{C}}$. Note however that all the components different from the $\mathcal{C}_i$ have genus zero and hence do not contribute to the sum. With these two inequalities we can analyze the degree of the image of the curve obtaining that

$$\begin{aligned}
\deg \varphi(\tilde{\mathcal{C}}) &= \sum_i \deg \varphi(\mathcal{C}_i) + \sum_j \deg \varphi(\mathcal{G}_j) + \\
&\quad + \sum_h \deg \varphi(\mathcal{D}_h) + \sum_k \deg \varphi(\mathcal{E}_k) \\
&\leq \sum_i A \max\{1,\ 2g(\mathcal{C}_i) - 2 + \#S_i\} + \sum_j 2 + \sum_h 3 + \sum_k 4 \\
&\leq A\{2[\sum_i g(\mathcal{C}_i) + \sum_j g(\mathcal{G}_j) + \sum_h g(\mathcal{D}_h) + \sum_k g(\mathcal{E}_k)] + \\
&\quad - 2\#\text{ comp } + \sum_i \#S_i + \sum_j 2 + \sum_h 3 + \sum_k 4\} \\
&\leq A\{2[\sum_i g(\mathcal{C}_i) + \sum_j g(\mathcal{G}_j) + \sum_h g(\mathcal{D}_h) + \sum_k g(\mathcal{E}_k)] + \\
&\quad - 2\#\text{ comp } + \sum_i \#S_i + \sum_j \#S_j + \sum_h \#S_h + \sum_k \#S_k\} \\
&\leq A\{2[\sum_i g(\mathcal{C}_i) + \sum_j g(\mathcal{G}_j) + \sum_h g(\mathcal{D}_h) + \sum_k g(\mathcal{E}_k)] + \\
&\quad - 2\#\text{ comp } + \#S + \#\text{ nodes }\} \\
&\leq A\{2[\sum_i g(\mathcal{C}_i) + \sum_j g(\mathcal{G}_j) + \sum_h g(\mathcal{D}_h) + \sum_k g(\mathcal{E}_k)] \\
&\quad - 2\#\text{ comp } + \#S + 2\#\text{ nodes }\} \\
&\leq A\{2g(\tilde{\mathcal{C}}) - 2 + \#S\} \\
&= A \max\{1, \chi(\tilde{\mathcal{C}} \setminus S)\},
\end{aligned}$$

where the last equality follows from the assumption that $\tilde{\mathcal{C}}$ has more than one irreducible component and from Lemma 7.1.5.

Step 2 Finally we prove the Theorem for a generic log stable map $\varphi : \tilde{\mathcal{C}} \to \mathbb{P}^2$ with $\varphi^{-1}(D) \subset S$. This time we divide the irreducible components of $\tilde{\mathcal{C}}$ in two groups

$$\tilde{\mathcal{C}} = \sqcup \mathcal{C}_i \sqcup \mathcal{P}_j,$$

where $\mathcal{P}_j$ are the components contracted to points by the map φ. From Step 1 we know that the restriction of φ to the irreducible components not contracted by φ verifies the conclusion of the Theorem, i.e., called $\mathcal{C}$ the union of all such components and $f = \varphi|_{\mathcal{C}}$ we have

$$\deg f(\mathcal{C}) \leq A \cdot \max\{1, \chi(\mathcal{C} \setminus S_{\mathcal{C}})\},$$

with $S_{\mathcal{C}} = S \cap \mathcal{C}$, provided that such a union of components is connected. If it is not connected is enough to consider each connected component separately. At the same time the degree of the image of $\tilde{\mathcal{C}}$ through φ is the same as $\deg f(\mathcal{C})$. Therefore the conclusion holds by the following computation:

$$\chi_S(\tilde{\mathcal{C}}) = 2g - 2 + \#S \geq 2g(\mathcal{C}) - 2 + \#S_{\mathcal{C}},$$

being the genus not less than the sum of the genus of its irreducible components, and being $S_{\mathcal{C}}$ contained in S.

□

Previous Theorem prove that the same result as Theorem 3.2.3 holds more generally for log-stable maps. In particular, image of log-stable maps is bounded by the Euler Characteristic. Moreover, it shows that the stability conditions gives a bound from below to the number of points of S in a component that maps over an irreducible component of the divisor D. At the same time a bound for the $S \cap \mathcal{P}_j$ is given, in such a way that for every contracted component P_j its Euler Characteristic is positive, for every genus $g(P_j)$.

7.2 Emptiness of moduli spaces

Our ultimate goal, as discussed in Chapter 5, is trying to extend these results to complements of divisors in $\mathbb{P}^2$ with less than three irreducible components. The idea we would like to formalize is the following:

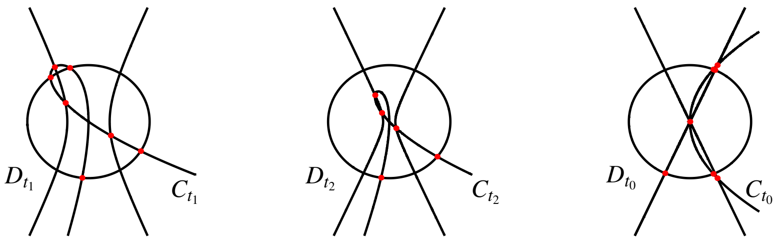


Figure 7.2: Idea of extension to lesser components.

Let D be a quartic in $\mathbb{P}^2$ which flattly deform to a conic and two lines. Consider such a deformation as a flat fibration over a connected base curve. If every curve in the fiber of D moves in the family it gives rise to a curve in the fiber of the conic and two lines. Such a curve should verify the bound of Theorem 7.1.6. If the curve has been moved keeping track of the intersection multiplicities with the divisors in each fiber then a similar bound holds also with respect to the divisor D in the starting fiber.

A picture of the previous idea can be seen in figure 7.2: here the algebraic family of divisors D_t is considered over an affine open of $\mathbb{P}^1$ and three fibers are drawn: D_{t_1}, D_{t_2} are quartics having two irreducible components while D_{t_s} is the conic and two lines appearing in Theorem 7.1.6.

From figure 7.2 one can see how the cardinality of the set S can a priori change in a deformation of the divisor D. We have already seen how to define intersection with D for stable curves with components mapping to an irreducible component of D: log stability condition gives a lower bound for the point of S in such irreducible component, even if the component is mapped over an irreducible component or it is contracted by the map. Now we want a way to control the set S with respect to a flat family of divisor. This in particular will cover the case of a two conic deforming to a conic and two lines, or even more generally a quartic deforming to a conic and two lines.

Instead of proving that $\#S$ can be controlled in a flat deformation we will prove that, given numerical data Γ, any class of log-stable maps with such data has constant $\#S$. Then we will see how properties of certain moduli space for maps with fixed numerical data Γ are related to Theorem 7.1.6.

7.2.1 Algebraic Hyperbolicity via moduli of log-stable maps

Let us look more carefully to the inequality of Theorem 7.1.6:

$$\deg f(\tilde{C}) \leq A \cdot \max\{1, \chi_S(\tilde{C})\}.$$

If we denote by β the curve class in $\mathbb{P}^2$ corresponding to $f(\tilde{C})$ we have that $\deg f(\tilde{C}) = \deg \beta$. Then if g and n denotes the genus of $\tilde{C}$ and the number of points of S we have that previous inequality can be rewritten as

$$\deg \beta \leq A \cdot \max\{1, 2g - 2 + n\};$$

in particular this implies that we cannot choose *independently* the quantities g, n and β for a map f. The first consequence of this reformulation is the following

Proposition 7.2.1. *Let $(\mathbb{P}^2, \mathcal{M}_D)$ be the DF pair given by considering the log structure induced by D. Let $\Gamma = (g, n, \beta, \vec{c})$ be discrete data such that*

$$\deg \beta \geq A \cdot \max\{1, 2g - 2 + n\}.$$

Then the moduli space $\mathcal{K}_\Gamma(\mathbb{P}^2, \mathcal{M}_D)$ is empty.

Proof. Every element of $\mathcal{K}_\Gamma(\mathbb{P}^2, \mathcal{M}_D)$ is a minimal log stable map $f : \tilde{C} \to \mathbb{P}^2$. By the description of Γ given in 6.3.8 at page 125, since

$$\beta \cdot c_1\big(\mathcal{O}_{\mathbb{P}^2}(D_i)\big) = \sum_{j=1}^{n} c_{ij},$$

with $D = D_1 + D_2 + D_3$ as usual, we see that $\#S < n$ (since some of the c_{ij} may a priori be zero). In these settings, Theorem 7.1.6 gives

$$\begin{aligned}\deg\beta = \deg f(\tilde{C}) &\leq A \cdot \max\{1, 2g-2+\#S\} \\ &\leq A \cdot \max\{1, 2g-2+n\}.\end{aligned}$$

□

In the spirit of previous Proposition we give the following

Definition 7.2.2. *In the settings above we call a discrete data* $\Gamma = (g, n, \beta, \vec{c})$ admissible *if*

$$\deg\beta \leq A \cdot \max\{1, 2g-2+n\}.$$

If $\vec{c}$ *as at least one element non zero for each* j*, we call* Γ strictly admissible. *We say that* Γ *is* A-admissible *if we want to explicitly specify the dependency from the constant* A.

One of the implication of the Proposition is that, given discrete data Γ, all the Γ-minimal log stable map in $\mathcal{K}_\Gamma(\mathbb{P}^2, \mathcal{M}_D)$ satisfy $\#S \leq n$ independently of $\vec{c}$. Moreover, for every vector of multiplicities $\vec{c}$ the set S is precisely the set of marked point P_α for which at least one of the $c_{i\alpha}$ is non zero. In particular, fixing Γ will automatically fix also $\#S$ for all the maps in $\mathcal{K}_\Gamma(\mathbb{P}^2, \mathcal{M}_D)$. We can rephrase this in the following

Corollary 7.2.3. *Theorem 7.1.6 implies that* $\mathcal{K}_\Gamma(\mathbb{P}^2, \mathcal{M}_D)$ *is empty for every vector of multiplicities* $\vec{c}$ *in* Γ*, for every non admissible discrete data* Γ*.*

Note that Proposition 7.2.1 and Corollary 7.2.3 are stated in the case in which D has three components however one could give a more general result, i.e. algebraic hyperbolicity of $\mathbb{P}^2 \setminus D$ for a simple normal crossing divisor D of any degree and any number of irreducible components implies the emptiness of the corresponding moduli spaces of log stable maps.

Now the deformation argument we sketched before can be applied in the following way: the properness of the stack

$\mathcal{K}_\Gamma(\mathbb{P}^2, \mathcal{M}_D)$ implies that the equality $\mathcal{K}_\Gamma(\mathbb{P}^2, \mathcal{M}_D) = \emptyset$ for all non admissible Γ extends to the general fiber of a flat deformation of the divisor D. Therefore, in order to prove algebraic hyperbolicity for the complement of a generic quartic in $\mathbb{P}^2$ we only need to show that the emptiness of $\mathcal{K}_\Gamma(\mathbb{P}^2, \mathcal{M}_D)$ for non admissible discrete data implies algebraic hyperbolicity.

Proposition 7.2.4. *Let $B > 0$ be an integer. Given a simple normal crossing divisor D in $\mathbb{P}^2$ denote by $\mathcal{M}_D$ the log structure associated. If for any discrete data Γ which is not B-admissible the moduli space $\mathcal{K}_\Gamma(\mathbb{P}^2, \mathcal{M}_D)$ is empty then every non-constant morphism $\tilde{f} : \tilde{C} \to \mathbb{P}^2$ verifies:*

$$\deg \tilde{f}(\tilde{C}) \leq B \cdot \max\{1, \chi_S(\tilde{C})\},$$

where $S = f^{-1}(D)$.

Proof. Let $\tilde{f} : \tilde{C} \to \mathbb{P}^2$, be a non-constant morphism. From Theorem 6.3.6 at page 124 it follows that there exists a minimal log map f over a geometric point having as underlying map f. To such a minimal map it corresponds a point in a moduli space $\mathcal{K}_\Gamma(\mathbb{P}^2, \mathcal{M}_D)$ for some $\Gamma = (g(\tilde{C}), n, f_*[\tilde{C}], \vec{c})$. Note that n here comes from the data associated to the prestable curve naturally defined by $\tilde{f}$, i.e. it does not depend on the log structure associated. In particular, if we fix the set of marked points as the set of pre-images of points of intersections $\tilde{f}(\tilde{C}) \cdot D$, it follows that $n = \#S$. Now the existence of the map f implies that $\mathcal{K}_\Gamma(\mathbb{P}^2, \mathcal{M}_D)$ is not empty. Therefore, by hypothesis Γ is B-admissible. Then

$$\begin{aligned}\deg \tilde{f}(\tilde{C}) = \deg \beta &\leq B \cdot \max\{1, 2g - 2 + n\} \\ &= B \cdot \max\{1, 2g - 2 + \#S\}.\end{aligned}$$

□

Given Proposition 7.2.1 and Proposition 7.2.4 one can state the following

Theorem 7.2.5. *Let D be a simple normal crossing divisor in $\mathbb{P}^2$ and let $\mathcal{M}_D$ be the log structure associated. Then $\mathbb{P}^2 \setminus D$ is algebraically hyperbolic if and only if there exists a positive constant A such that for every discrete data Γ not A-admissible the moduli space $\mathcal{K}_\Gamma(\mathbb{P}^2, \mathcal{M}_D)$ is empty.*

7.3 Properness of the stack

Theorem 7.2.5 gives a sort of "moduli interpretation" of algebraic hyperbolicity, translating the problem to the emptiness of some moduli space of logarithmic stable maps. In particular in the three components case, Corvaja and Zannier result together with its generalization, Theorem 7.1.6, can be read as a property of such moduli spaces, providing emptiness for non A-admissible discrete data with constant $A = 2^{15} \cdot 35$. Moreover, the emptiness of the moduli space is a property that behaves well with respect to flat deformation of the divisor D. In particular, the following property of $\mathcal{K}_\Gamma(\mathbb{P}^2, \mathcal{M}_D)$ holds:

Proposition 7.3.1. *Let $D, \mathcal{M}_D$ as before. Then there is a proper map*

$$\mathcal{K}_\Gamma(\mathbb{P}^2, \mathcal{M}_D) \to \mathcal{M}(D),$$

where $\mathcal{M}(D)$ is the moduli of divisor D in $\mathbb{P}^2$ of fixed degree.

Previous proposition follows from Theorem 6.3.10 and the fact that D here is assumed (simple) normal crossing, hence in particular toroidal. Now giving a deformation of the divisor $D^{(3)} = D_1 + D_2 + D_3$ one gets that the properness of the map allows to extend the emptiness of the stack $\mathcal{K}_\Gamma(\mathbb{P}^2, \mathcal{M}_{D^{(3)}})$ to the generic member of the deformation. In particular this gives emptiness of the corresponding stack for the generic quartic degenerating to the divisor consisting of a conic and two lines. The union of all such conditions over all possible genus, number of marked points and degree of the push-forward of the fundamental class of a curve, gives algebraic hyperbolicity for the complement of a very generic quartic. Therefore the following theorem holds:

Theorem 7.3.2. *Let D be a very generic quartic in $\mathbb{P}^2$. If D has only simple normal crossing singularities then $\mathbb{P}^2 \setminus D$ is (weakly) algebraic hyperbolic.*

Here by very generic we mean a part from countably many exception, i.e. in the complement of a countable union of proper closed subvarieties in the Hilbert scheme of plane quartics, $Hilb^2(4t-2) = Proj(H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(4)))$. Note that the only interesting part of previous Theorem is given by the cases of log irregularity strictly smaller than two; the other cases follows from Theorem 3.1.1 and Theorem 3.2.3. Moreover the constant appearing in the hyperbolicity condition here is the same as the one given by Corvaja and Zannier Theorem. in particular if a different strategy were considered, i.e. extending the four lines' case instead of the three components' one, the constant could have been lowered also for the generic quartic with three components. The rest of the chapter will be devoted to the proof of Theorem 7.3.2.

We stress that the very generic hypothesis will allow to drop the simple normal crossing requirement, since a very generic quartic can be always be chosen to have normal crossing singularities. However we still prefer to explicitly state it because the result on the properness of the stack requires it as a fundamental hypothesis. Present work by Abramovich, Chen and others seems to give the possibility to obtain the properness of the stack dropping this hypothesis and then making it redundant also in our context.

The main step in the proof of Theorem 7.3.2 is to prove that being algebraic hyperbolic does not depend on the number of irreducible components of the divisor D. Once such independence is proved the result will follow from the extensions of Corvaja and Zannier's Theorem 3.2.3. This invariance will be proved using the moduli interpretation of hyperbolicity given by Theorem 7.2.5 and using the fact that for the generic member of a deformation to a conic and two lines the stack $\mathcal{K}_\Gamma(\mathbb{P}^2, \mathcal{M}_D)$ coincide with the same stack in the special fiber. Hence the emptiness of these moduli spaces in the case of three components extends to the case of very generic quartic. Then the equivalence between the emptiness of the stacks for non admissible discrete data and algebraic hyperbolicity will finish the proof.

7.3.1 Invariance under deformation

In this subsection we focus on the following

Proposition 7.3.3. *Let D be the generic member of a flat deformation of a quartic into $D^{(3)}$, a divisor consisting of a conic and two lines. Then for any discrete data Γ one has*

$$\mathcal{K}_\Gamma(\mathbb{P}^2, \mathcal{M}_D) = \mathcal{K}_\Gamma(\mathbb{P}^2, \mathcal{M}_{D^{(3)}}),$$

where $\mathcal{M}_D$ and $\mathcal{M}_{D^{(3)}}$ are the log structure associated to D and $D^{(3)}$ respectively.

Proof. The only real issue in the previous statement is that a flat deformation of a quartic is not in general log smooth. Thus the total space in the log category does not automatically give a log map which allows to use the proper map of Proposition 7.3.1. However, since all the log structures involved come from normal crossing divisor, it is enough to consider a normal crossing divisor in the total space such that, its restriction to the special fiber gives $D^{(3)}$, and the restriction to the generic fiber gives D.

Consider a flat deformation $h : \mathcal{D} \to B$ of quartics such that $\mathcal{D}_b = D$ for the generic point $b \in B$, and $\mathcal{D}_s = D^{(3)}$ for a close point $s \in B$. Since all the members of $\mathcal{D}$ are plane quartics the total space $\mathcal{D}$ naturally embeds in $\mathbb{P}^2 \times B$. Moreover, without loss of generality we can assume B to be smooth projective of dimension 1, or even more, we can choose B such that its completion is contained in $\mathbb{P}^1$ (one could also assumed B to be the spectrum of a local ring in which s is the only closed point and b is the generic one). Notice that such a deformation always exists because we can consider the line in the Hilbert Scheme of quartics containing the conic and two lines. The affine open set in the line which does not contain any (too) singular quartic will give the desired deformation.

Now we claim that $\mathcal{D} \subset \mathbb{P}^2 \times B$ is a normal crossing divisor in $\mathbb{P}^2 \times B$. First we note that, up to restricting B if necessary, we may assume that $\mathcal{D}$ has normal crossing singularities. Finally $\mathcal{D}$ can be chosen to be defined locally by one equation depending on the uniformizing parameter of B. Such equations defines $\mathcal{D}$ inside $\mathbb{P}^2 \times B$. Here we used the fact that we can choose a generic normal crossing $D^{(3)}$ in any way since Theorem 7.1.6 applies to every degree 4 and three component plane curve, provided it has normal crossing singularities.

From the fact that $\mathcal{D}$ is a normal crossing divisor we have that $\mathbb{P}^2 \times B$ has a natural fs log structure $\mathcal{M}$ with the property that on each fiber $\mathbb{P}^2 = \mathbb{P}^2_b$ of the second projection the restriction of $\mathcal{M}$ gives the log structure defined by the normal crossing divisor $\mathcal{D}_b$. This exhibits $\mathbb{P}^2 \times B$ as a log scheme with a log map to B. Such map has a flat underlying map as a map of schemes. Moreover, being $\mathcal{D} \to B$ flat, it gives a log map $\mathbb{P}^2 \times B$, with B with trivial log structure.

Now Theorem 6.3.10 together with Proposition 7.3.1 applied to $\mathbb{P}^2 \times B$ gives the conclusion. □

Example 7.3.4. Considering the case of two conics, and a flat family given by a constant irreducible conic $\mathcal{C}_1$ given by $f = 0$ and a conic $\mathcal{C}_{2,t}$ degenerating to two lines as

$$x^2 - y^2 = t$$

over $B = Spec\,\mathbb{C}[t]$. On $\mathbb{P}^2 \times B$ the divisor D given by $x^2 - y^2 - t = 0$ and $f = 0$ is a normal crossing divisor provided that $\mathcal{C}_1 \cap \mathcal{C}_{2,t}$ intersect transversally. Up to reducing B we can always assume this to hold. Then over $\mathbb{P}^2 \times B$ the log structure given by D is fs and the restriction to every fiber $\mathbb{P}^2_t$ is the log structure given by $\mathcal{C}_1 + \mathcal{C}_{2,t}$.

7.3.2 Properness of the moduli space of log-stable maps

The end of the proof of Theorem 7.3.2 now follows from the results proved in the previous sections.

Proof of Theorem 7.3.2. Let D be a plane quartic and denoted by $D^{(3)}$ a conic and two lines in general position. By Theorem 7.2.5 we only need to show that there exists a constant A' such that $\mathcal{K}_\Gamma(\mathbb{P}^2, \mathcal{M}_D)$ is empty for all non A'-admissible discrete data Γ. By Proposition 7.3.3 we have that $\mathcal{K}_\Gamma(\mathbb{P}^2, \mathcal{M}_D)$ coincide with $\mathcal{K}_\Gamma(\mathbb{P}^2, \mathcal{M}_{D^{(3)}})$. At the same time, by Corvaja and Zannier's result, Theorem 3.2.3, and by Proposition 7.2.1, the stack $\mathcal{K}_\Gamma(\mathbb{P}^2, \mathcal{M}_{D^{(3)}})$ is empty for all non admissible Γ with respect to the constant $A = 2^{15} \cdot 35$. Then the same holds for the stack with respect to the log structure coming from D finishing the proof. In particular this gives vanishing for the stack with non A-admissible discrete data, and hence algebraic hyperbolicity, with the same constant appearing in Theorem 7.1.6. $\square$

Remark 7.3.5. We stress that the very generic hypothesis here is necessary since for every discrete data Γ the emptiness of the stack extends to the *generic* member of the deformation. This is equivalent to assume that for every Γ the result holds away from a finite number of quartics. At the same time to prove algebraic hyperbolicity we need to consider the moduli space $\mathcal{K}_\Gamma(\mathbb{P}^2, \mathcal{M}_D)$ for infinitely many, although countably many, discrete data. This implies that the result holds away from countably many closed subset of the parameter space of plane quartics.

The discreteness of the Γ follows from the fact that

$$\beta \in H^2(\mathbb{P}^2, \mathbb{Z}) \simeq \mathbb{Z}$$

and given g, n and β the choices of possible contact orders $\vec{c}$ are finite.

In particular Theorem 7.3.2 gives a weaker result than Theorem 7.1.6 for complements of $\mathbb{P}^2$ in which the log irregularity is less than 2. While it uses in an essential way the fact the Corvaja and Zannier's result, as well as its extensions, applies to a generic quartic with three irreducible components, our result gives genericity only for a fixed discrete data Γ but cannot recover such genericity for proving algebraic hyperbolicity.

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